When you are done with your homework you should be able to…

- π Determine continuity at a point and continuity on an open interval
- π Determine one-sided limits and continuity on a closed interval
- π Use properties of continuity
- π Understand and use the Intermediate Value Theorem

DEFINITION OF CONTINUITY

CONTINUITY AT A POINT: A function *f* is **continuous at *c*** if the following **three** conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

CONTINUITY ON AN OPEN INTERVAL:

A function is **continuous on an open interval (*a*,*b*)** if it is continuous at **each point in the interval**. A function that is continuous on the entire real line is **everywhere continuous**.

1. Draw the graph of the following functions with the given characteristics on the open interval from *a* to *b*:
   a. The function has a removable discontinuity at $x = 0$

   b. The function has a nonremovable discontinuity at $x = 0$

THEOREM: THE EXISTENCE OF A LIMIT

Let *f* be a function and let *c* and *L* be real numbers. The limit of $f(x)$ as *x* approaches *c* is *L* if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

CONTINUITY ON A CLOSED INTERVAL:

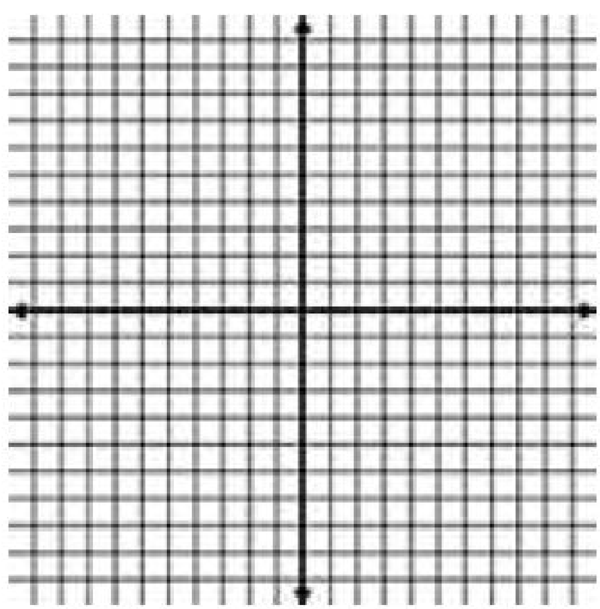
A function *f* is **continuous on the closed interval [*a*,*b*]** if it is continuous on the open interval (*a*, *b*) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

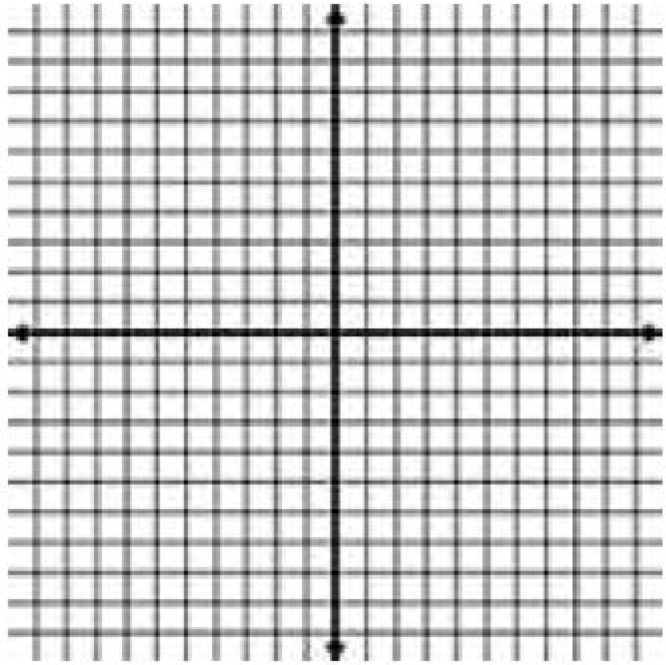
The function *f* is **continuous from the right at *a*** and **continuous from the left at *b***.

2. Graph each function and use the definition of continuity to discuss the continuity of each function.

   a. $f(x) = \dfrac{x^2 - 4}{x + 2}$

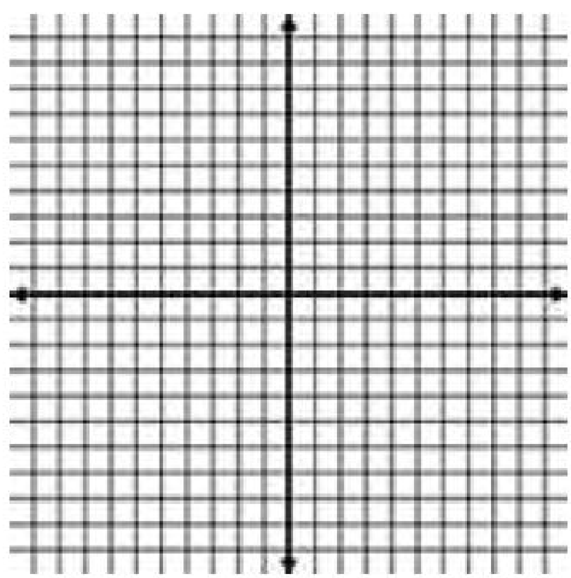


b. $g(x) = \tan x$



c.

$$y = \begin{cases} |x|, & x \le 2 \\ -x, & 2 < x < 4 \\ \dfrac{x^2}{4}, & x \ge 4 \end{cases}$$



THEOREM: PROPERTIES OF CONTINUITY

If *b* is a real number and *f* and *g* are continuous at $x = c$ then the following functions are also continuous at *c*.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g},\ g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + x_0$
2. Rational functions: $r(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$
4. Trigonometric functions: $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x$

3. Explain why the following functions are continuous at every point in their domains.

   a. $f(x) = \sqrt{x} - \tan x$

   Properties used:

b. $f(x)=\dfrac{5-x}{x\sin x}$

Properties used:

THEOREM: CONTINUITY OF A COMPOSITE FUNCTION

If *g* is continuous at *c* and *f* is continuous at $g(c)$ then the composite function $(f\circ g)(x)=f(g(x))$ is continuous at *c*.

4. Explain why the following functions are continuous at every point in their domains.

a. $f(x)=\sqrt[3]{x^2-8x+1}$

Properties used:

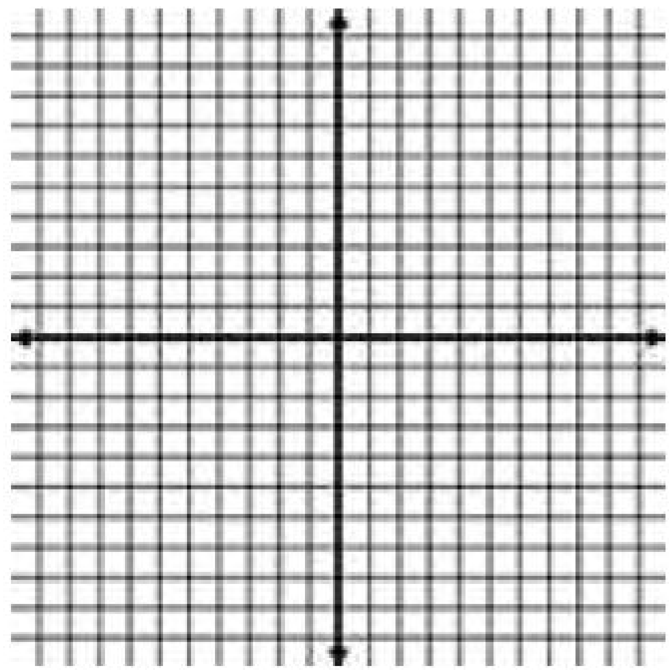
b. $g(x)=\tan\left(2x^2\right)$

Properties used:

THEOREM: THE INTERMEDIATE VALUE THEOREM

If *f* is continuous on the closed interval $[a,b]$ and *k* is any number between $f(a)$ and $f(b)$, then there is **at least** one number *c* in $[a,b]$ such that $f(c)=k$.

5. Consider the function $f(x)=\cos 2x$ on the closed interval $[0,\pi]$.
   a. Sketch the graph of *f* by hand.



   b. State the reason why we can apply the intermediate value theorem (IVT).

   c. Use the IVT to find *c* such that $f(c)=\frac{1}{2}$.