When you are done with your homework you should be able to…

- π Evaluate a limit using properties of limits
- π Develop and use a strategy for finding limits
- π Evaluate a limit using dividing out and rationalizing techniques
- π Evaluate a limit using the Squeeze Theorem

DIRECT SUBSTITUTION

If the limit of $f(x)$ as x approaches c is $f(c)$, then the limit may be evaluated using **direct substitution**. That is, $\lim_{x \to c} f(x) = f(c)$. These types of functions are continuous at *c*.

THEOREM: SOME BASIC LIMITS

Let *b* and *c* be real numbers and let *n* be a positive integer.

LIMIT	EXAMPLE
1. $\lim_{x \to c} b = b$	
2. $\lim_{x \to c} x = c$	
3. $\lim_{x \to c} x^n = c^n$	

THEOREM: PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \qquad \text{and} \qquad \lim_{x\to c} g(x) = K$$

LIMIT

1. Scalar multiple: $\lim_{x\to c}\left[bf(x)\right] = bL$
2. Sum or difference: $\lim_{x\to c}\left[f(x)\pm g(x)\right] = L\pm K$
3. Product: $\lim_{x\to c}\left[f(x)g(x)\right] = LK$
4. Quotient: $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K},\ K\neq 0$
5. Power: $\lim_{x\to c}\left[f(x)\right]^n = L^n$

1. Find the limit. Identify the individual functions and the properties you used to evaluate the limit.

   a. $\lim_{x\to 2}\left(5-x^2\right)$

   $f(x) =$ ________, $g(x) =$ __________

   Properties used:

b. $\lim_{x\to 4}\left(x\left(2-9x^2\right)\right)$

$f(x)=$ ________, $g(x)=$ __________

Properties used:

c. $\lim_{x\to 0}\left(\frac{6x-5}{x^3-2}\right)$

$f(x)=$ ________, $g(x)=$ __________

Properties used:

THEOREM: LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If *p* is a polynomial and *c* is a real number, then

$$\lim_{x\to c} p(x)=p(c).$$

If *r* is a rational function given by $r(x)=p(x)/q(x)$ and *c* is a real number such that $q(c)\neq 0$, then

$$\lim_{x\to c} r(x)=r(c).$$

2. Find the following limits.

a. $\lim_{x\to 3}\left(-x^4+6x^2-2\right)$

b. $\lim_{x \to -4} \frac{x^3 - 1}{2x + 7}$

THEOREM: THE LIMIT OF A RADICAL FUNCTION

Let *n* be a positive integer. The following limit is valid for all *c* if *n* is odd, and is valid for $c > 0$ if *n* is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

3. Find the following limits.

a. $\lim_{x \to 225} \sqrt{x}$

b. $\lim_{x \to -243} \sqrt[5]{x}$

THEOREM: THE LIMIT OF A COMPOSITE FUNCTION

Let *f* and *g* be functions with the following limits.

$$\lim_{x \to c} g(x) = L \quad \text{and} \quad \lim_{x \to L} f(x) = f(L)$$

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

4. Find the following limits:

a. $\lim\limits_{x\to 10}(x-1)$

b. $\lim\limits_{x\to 9}\left(\sqrt{x}\right)$

c. $\lim\limits_{x\to 10}\left(\sqrt{x-1}\right)$

THEOREM: LIMITS OF TRIGONOMETRIC FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

1. $\lim\limits_{x\to c}(\sin x)=\sin c$
2. $\lim\limits_{x\to c}(\cos x)=\cos c$
3. $\lim\limits_{x\to c}(\tan x)=\tan c$
4. $\lim\limits_{x\to c}(\cot x)=\cot c$
5. $\lim\limits_{x\to c}(\sec x)=\sec c$
6. $\lim\limits_{x\to c}(\csc x)=\csc c$

5. Evaluate the following limits.

a. $\lim\limits_{x\to \frac{2\pi}{3}}(\tan x)$

b. $\lim\limits_{x\to \pi}(\sin x)$

c. $\lim\limits_{x\to \pi}\left(\cos\frac{x}{6}\right)$

THEOREM: FUNCTIONS THAT AGREE AT ALL BUT ONE POINT

Let *c* be a real number and let $f(x)=g(x)$ for all $x \neq c$ in an open interval containing *c*. If the limit of $g(x)$ as *x* approaches *c* exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

6. Find the following limits. What happens when you try direct substitution? If you use the theorem above, be sure to identify $g(x)$.

   a. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

   b. $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$

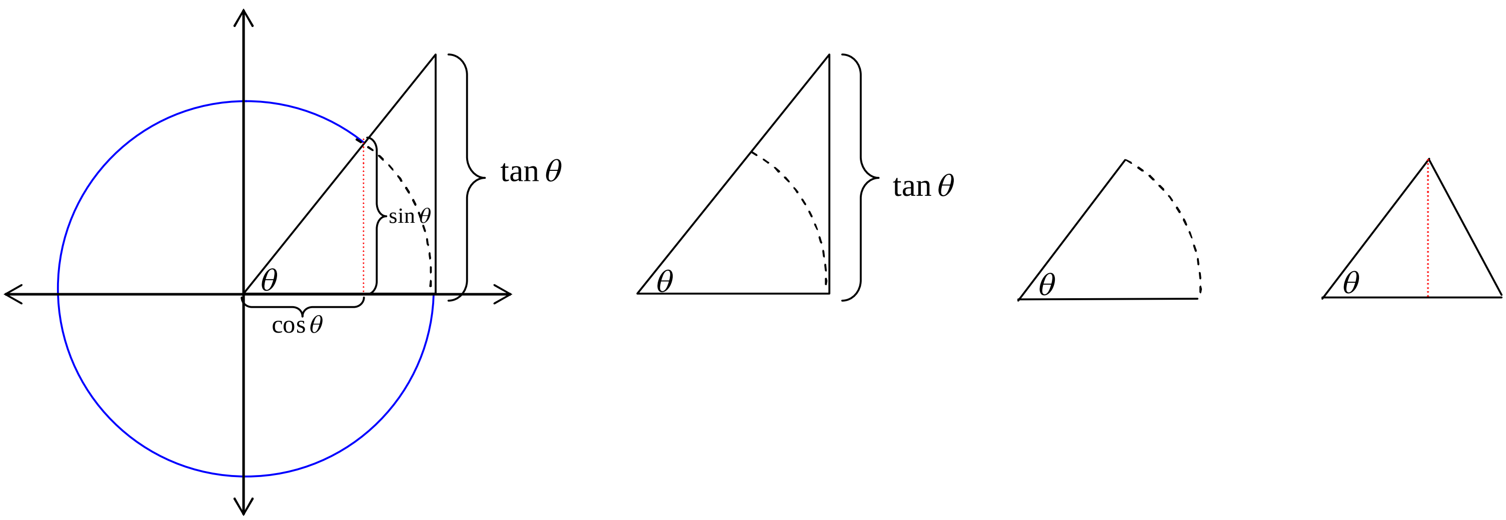
THE SQUEEZE THEOREM

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ then $\lim_{x\to c} f(x)$ exists and is equal to L.

THEOREM: TWO SPECIAL TRIGONOMETRIC LIMITS

1. $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

2. $$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$



7. Find the following limits.

a. $\lim_{x\to 0} \frac{\tan^2 x}{x}$

b. $\lim_{x\to \pi/4} \frac{1-\tan x}{\sin x - \cos x}$

STRATEGIES FOR FINDING LIMITS

1. Try using direct substitution first. If this works, you are done! If not, go to step 2.
2. If you obtain an indeterminate result when using direct substitution (0/0), try
   a. Factoring the numerator and denominator, dividing out common factors, and then use direct substitution on the new expression.
   b. Rationalizing the numerator, and then use direct substitution on the new expression.
3. If you obtain an indeterminate result when using direct substitution (0/0), on a trigonometric expression try
   a. Rewriting the expression using trigonometric identities, and then use direct substitution on the new expression.
   b. Rewriting the expression using trigonometric identities, and then use the special trigonometric limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$
4. Verify your result by graphing the function on your graphing calculator.

8. Find the following limits.

a. $\lim_{x\to 4}\frac{\sqrt{x-2}-\sqrt{2}}{x-4}$

b. $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)^2-2x^2}{\Delta x}$

c. $\lim_{x\to 0}\frac{\frac{1}{x-3}+\frac{1}{3}}{x}$

d. $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}$