When you are done with your homework you should be able to…

- π Develop properties of the natural exponential function
- π Differentiate natural exponential functions
- π Integrate natural exponential functions

Warm-up:

1. Differentiate the following functions with respect to x.

a. $y = x^{5x}$

$$\ln y = \ln x^{5x}$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[(5x)(\ln x)\right]$$

$$\cancel{y}\,\frac{dy/dx}{\cancel{y}} = \left(5\ln x + 5\cancel{x}\left(\frac{1}{\cancel{x}}\right)\right)y$$

$$\boxed{\frac{dy}{dx} = 5x^{5x}(1+\ln x)}$$

b. $f(x) = \ln e^{\cos 2x}$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\cos 2x$$

$$\boxed{f'(x) = -2\sin 2x}$$

DEFINITION: THE NATURAL EXPONENTIAL FUNCTION

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$g(x) = e^x \qquad g(x) = f^{-1}(x)$$

That is,

$$f(g(x)) = x = g(f(x))$$

The inverse relationship between the natural logarithmic function and the natural exponential function can summarized as follows:

$$y = \ln x \iff e^y = x$$

Example 1: Solve the following equations. Give the **exact result** and then round to 3 decimal places.

a. $e^{\ln 6x} = 20$

$6x = 20$

$x = \left\{\frac{10}{3}\right\}$

b. $\dfrac{5000}{1+e^{2x}} = \dfrac{2}{1}$

$5000 = 2(1+e^{2x})$

$2500 = 1 + e^{2x}$

$\ln e^{2x} = \cancel{\ln} 2499$

$2x = \ln 2499$

$x = \frac{1}{2}\ln 2499$ exact

$x = \boxed{\left\{\ln\sqrt{2499}\right\}}$ exact

$x \approx 3.91$ approximate

c. $\ln 8x = 3 \iff e^3 = 8x$

$$x = \left\{\frac{e^3}{8}\right\}$$

$$x \approx 2.51$$

d. $e^{2x} - 2e^x - 8 = 0$

$$(e^x)^2 - 2(e^x) - 8 = 0$$

$$(e^x - 4)(e^x + 2) = 0$$

$$e^x - 4 = 0 \quad \text{or} \quad e^x + 2 = 0$$

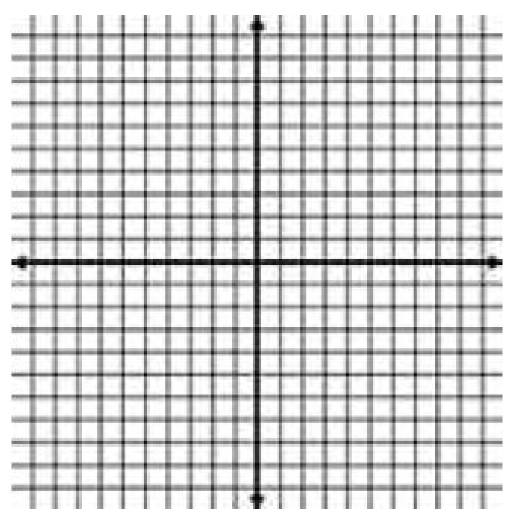
$$e^x = 4 \quad \text{or} \quad \cancel{e^x = -2}$$

extraneous

$$\ln e^x = \ln 4$$

$$x = \boxed{\{\ln 4\}}$$

Example 2: Sketch the graph of $f(x) = 2e^{x-1}$ without using your graphing calculator.



DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$g(t) = \left(e^{-2t} - e^{+t}\right)^3$

$g'(t) = 3\left(e^{-2t} - e^{t}\right)^2 \left(e^{-2t}(-2) - e^{t}\right)$ leave the clean-up to you!

THEOREM: DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}\left[e^x\right] = e^x$

2. $\frac{d}{dx}\left[e^u\right] =$ $e^u \cdot du/dx$

Example 3: Find the derivative with respect to x.

a. $\frac{d}{dx}(y) = \frac{d}{dx}\left(e^{5-x^3}\right)$

$y' = e^{5-x^3} \frac{d}{dx}\left(5-x^3\right)$

$\boxed{y' = -3x^2 e^{5-x^3}}$

b. $f(x) = xe^{3x}$

$f'(x) = 1e^{3x} + xe^{3x} \cdot 3$

$\boxed{f'(x) = e^{3x}(1+3x)}$

c. $y=\ln\frac{1+e^x}{1-e^x}$

$\frac{d}{dx}y=\frac{d}{dx}\ln(1+e^x)-\frac{d}{dx}\ln(1-e^x)$

$y'=\frac{e^x}{1+e^x}-\frac{-e^x}{1-e^x}$

$y'=\frac{e^x(1-e^x)+e^x(1+e^x)}{1^2-(e^x)^2}$

$y'=\frac{e^x[(1-e^x)+(1+e^x)]}{1-e^{2x}}$

$\boxed{y'=\frac{2e^x}{1-e^{2x}}}$

d. $y=\frac{e^x-e^{-x}}{2}$

e. $e^{xy} + x^2 - y^2 = 10$

f. $F(x) = \int_0^{e^{2x}} \ln(t+1)\,dt$

$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$

$\frac{d}{dx} F(x) = \frac{d}{dx} \int_0^{e^{2x}} \ln(t+1)\,dt$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$u = e^{2x}$

$\frac{du}{dx} = 2e^{2x}$

$F'(x) = \left[\frac{d}{du} \int_0^u \ln(t+1)\,dt\right] \frac{du}{dx}$

$F'(x) = [\ln(u+1)]\, 2e^{2x}$

$\boxed{F'(x) = 2e^{2x} \ln(e^{2x}+1)}$

Example 4: Find an equation of the tangent line of the function $1+\ln xy = e^{x-y}$ at the point $(1,1)$.

$$\frac{d}{dx}\left(1+\ln xy\right) = \frac{d}{dx}\left(e^{x-y}\right)$$

$$\frac{1}{xy}\left(1y + x\frac{dy}{dx}\right) = \left(e^{x-y}\right)\left(1-\frac{dy}{dx}\right)$$

$$\frac{1}{(1)(1)}\left((1)+(1)\frac{dy}{dx}\right) = e^{1-1}\left(1-\frac{dy}{dx}\right)$$

$$1+\frac{dy}{dx} = 1-\frac{dy}{dx}$$

$$2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0$$

$$\boxed{y=1}$$

$$y-1 = 0(x-1)$$

$$y = 1$$

Example 5: Find the extrema and points of inflection of the function

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}.$$

$$g'(x) = \frac{1}{\sqrt{2\pi}}\left(e^{-(x-3)^2/2}\right)\left(\frac{-1}{\cancel{2}}\cdot \cancel{2}(x-3)^1(1)\right)$$

$$g'(x) = \frac{(3-x)e^{-(x-3)^2/2}}{\sqrt{2\pi}}$$

$$0 = 3-x$$

$$x = 3$$

$$c = 3$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}\left[(-1)e^{-(x-3)^2/2} + (3-x)^2 e^{-(x-3)^2/2}\right]$$

$$g''(x) = \frac{-e^{-(x-3)^2/2}\left[1-(3-x)^2\right]}{\sqrt{2\pi}}$$

$$0 = 1-(3-x)^2$$

$$\sqrt{(3-x)^2} = \sqrt{1}$$

$$3-x = \pm 1$$

$$3-x = 1 \qquad x = 2$$

$$3-x = -1 \qquad x = 4$$

$$g''(x) = \frac{-e^{-(x-3)^2/2}}{\sqrt{2\pi}}\left[1-(3-x)^2\right]$$

x	0	2	3	4	5
sign of g''	+		−		+

$g''(0) =$ positive

$g''(3) =$ negative

$g''(5) =$ positive

2nd der. test

$$g''(3) = -\frac{e^{-(x-3)^2/2}}{\sqrt{2\pi}} < 0$$

$$\boxed{\text{relative max at } (3, g(3)) = \left(3, \tfrac{1}{\sqrt{2\pi}}\right)}$$

P.O.I.: $(2, g(2)) = \left(2, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right)$

$(4, g(4)) = \left(2, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right)$

THEOREM: INTEGRATION RULES FOR NATURAL EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

1. $\int e^x dx = e^x + C$

2. $\int e^u du = e^u + C$

Example 6: Find the indefinite integrals and evaluate the definite integrals.

a. $\int e^{12x} dx = \int e^u \frac{du}{12} = \boxed{\frac{1}{12} e^{12x} + C}$

$= \frac{1}{12} \int e^u du$

$= \frac{1}{12}(e^u + C_1)$

$u = 12x$

$\frac{du}{dx} = 12$

$dx = \frac{du}{12}$

b. $\int x^4 e^{1-x^5} dx = \int \cancel{x^4} e^u \left(-\frac{du}{5\cancel{x^4}}\right)$

$= -\frac{1}{5}\int e^u du$

$= -\frac{1}{5} e^u + C$

$= \boxed{-\frac{1}{5} e^{1-x^5} + C}$

$u = 1 - x^5$

$\frac{du}{dx} = -5x^4$

$dx = -\frac{du}{5x^4}$

more complicated than e^{2x}

c. $\int \frac{e^{2x}}{1+e^{2x}} dx = \int e^{2x}(1+e^{2x})^{-1} dx$

$= \int \cancel{e^{2x}} u^{-1} \left(\frac{du}{2\cancel{e^{2x}}}\right)$

$= \frac{1}{2}\int u^{-1} du$

$= \frac{1}{2} \cdot \ln|u| + C$

$= \frac{1}{2}\ln(1+e^{2x}) + C$

$= \boxed{\ln\sqrt{1+e^{2x}} + C}$

$u = 1 + e^{2x}$

$\frac{du}{dx} = 2e^{2x}$

$dx = \frac{du}{2e^{2x}}$

d. $\int \frac{e^{2x}+2e^{x}+1}{e^{x}}dx = \int\left[\frac{(e^x)^2}{e^x}+\frac{2e^x}{e^x}+\frac{1}{e^x}\right]dx$

$= \int (e^x + 2 + e^{-x})dx$

$= e^x + 2x + -e^{-x} + C$

$= \boxed{e^x + 2x - e^{-x} + C}$

$u = -x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int e^{-x} dx$

$= \int e^u (-du)$

$= -\int e^u du$

$= -e^u + C$

$= -e^{-x} + C$

e. $\int e^{\tan 2x} \sec^2 2x dx$

try the power 1st for the u-sub

$u = \tan 2x$

$\frac{du}{dx} = 2\sec^2 2x$

$dx = \frac{du}{2\sec^2 2x}$

$= \int e^u \cancel{\sec^2 2x}\left(\frac{du}{2\cancel{\sec^2 2x}}\right)$

$= \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^u + C$

$= \boxed{\frac{1}{2}e^{\tan 2x} + C}$

f. $\int_0^1 \frac{e^x}{5-e^x}dx = \int_0^1 e^x (5-e^x)^{-1} dx$ more complicated

$u = 5-e^x$

$\frac{du}{dx} = -e^x$

$dx = -\frac{du}{e^x}$

upper: $5-e^1 = 5-e$

lower: $5-e^0 = 4$

$= \int_4^{5-e} \cancel{e^x} u^{-1} \left(-\frac{du}{\cancel{e^x}}\right)$

$= -\int_4^{5-e} u^{-1} du$

$= -\ln|u| \Big|_4^{5-e}$

$= -(\ln|5-e| - \ln 4)$

$= \boxed{-\ln \frac{5-e}{4}} = \boxed{\ln \frac{4}{5-e}}$

Example 7: Solve the differential equation.

$dx \frac{dy}{\cancel{dx}} = (e^x - e^{-x})^2$ dx typo

$\int dy = \int (e^x - e^{-x})^2 dx$

$y = \int ((e^x)^2 - 2(e^x)(e^{-x}) + (e^{-x})^2) dx$

$\frac{1}{2}\int 2e^{2x} dx$

$y = \int (e^{2x} - 2 + e^{-2x}) dx$

$y = \frac{1}{2}e^{2x} - 2x + \left(-\frac{1}{2}\right)e^{-2x} + C$

$\boxed{y = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C}$

Example 8: The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation $\int_0^x 0.3e^{-0.3t}dt = \frac{1}{2}$. Solve the equation.

$\int f(g(x))g'(x)\,dx$

$= F(g(x)) + C$

$-\int_0^x 0.3e^{-0.3t}\,dt = \frac{1}{2}$

$-\left(e^{-0.3t}\Big|_0^x\right) = \frac{1}{2}$

$-\left(e^{-0.3x} - e^{(-0.3)(0)}\right) = \frac{1}{2}$

$-e^{-0.3x} + 1 = \frac{1}{2}$

$-e^{-0.3x} = -\frac{1}{2}$

$\ln e^{-0.3x} = \ln\frac{1}{2}$

$-0.3x = \ln 1 - \ln 2$

$-0.3x = -\ln 2$

$x = \frac{\ln 2}{0.3}$

$x \approx 2.31$

$.31\ \text{min} \times \frac{60\ \text{sec}}{\text{min}} = 18.6\ \text{sec}$

$\boxed{\text{The median waiting time for the customers is approximately 2 minutes, 19 seconds}}$