MATH 1250/GRACEY
WORKSHEET/5.4-5.5

NAME____

$$\ln\left(e^x\right) = x \quad \text{and} \quad e^{\ln x} = x$$

Properties of Logarithmic Functions to Base a

1. $\log_a 1 = 0$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_2 x = 5$
$\Leftrightarrow 2^5 = x$
$x = 32$

Properties of Inverse Functions

1. $y = a^x \Leftrightarrow x = \log_a y$
2. $a^{\log_a x} = x,\ x > 0$
3. $\log_a a^x = x,\ \forall x$

1. Evaluate without using a calculator.

a. $\log_6 36 = \log_6 6^2 = \boxed{2}$

b. $\log_3 \frac{1}{81} = \log_3 3^{-4} = \boxed{-4}$

2. Write the exponential equation as a logarithmic equation or vice versa.

a. $\log_2 256 = 8 \Leftrightarrow \boxed{2^8 = 256}$

b. $49^{1/2} = 7$ $\Leftrightarrow$ $\log_{49} 7 = \frac{1}{2}$

3. Solve the equation. Round to the nearest thousandth.

a. $3\left(5^{x-1}\right) = 86$

$\frac{3(5^{x-1})}{3} = \frac{86}{3}$

$\ln 5^{x-1} = \ln \frac{86}{3}$

$\frac{(x-1)\ln 5}{\ln 5} = \frac{\ln \frac{86}{3}}{\ln 5}$

$x - 1 = \frac{\ln 86/3}{\ln 5}$

$x = \frac{\ln 86/3}{\ln 5} + 1$

$x = \frac{\ln 86/3 + \ln 5}{\ln 5}$

$x = \frac{\ln\left(\frac{430}{3}\right)}{\ln 5} \approx 3.09$

b. $\log\left(\sqrt{x-5}\right) = 4.8$

common log-base is 10

$\frac{1}{2}\log(x-5) = 4.8$

$\log(x-5) = 9.6$

$10^{9.6} = x - 5$

$x = 5 + 10^{9.6}$

$10^{9.6} = 1 \times 10^{9.6}$

c. $(\ln x)^2 - \ln x^3 = 0$

$(\ln x)^2 = (\ln x)(\ln x)$

$\ln x^3 = 3 \ln x$

$(\ln x)^2 - 3(\ln x) = 0$

$(\ln x)\left[\ln(x) - 3\right] = 0$

$\ln x = 0$ or $\ln x - 3 = 0$

$e^0 = x$ $\quad$ $\ln x = 3$

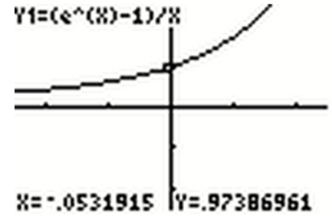
$x = 1$ $\quad$ $e^3 = x$

$x = e^3$

$\{1, e^3\}$

$f(x) = e^x$; $f'(x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$f'(x) = \lim_{\Delta x \to 0} e^x e^{\Delta x}$



- DERIVATIVES OF EXPONENTIAL FUNCTIONS

Theorem: Derivative of the Natural Exponential Function
Let u be a differentiable function of x.

1. $$\frac{d}{dx}\left[e^x\right] = e^x$$

2. $$\frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$$

4. Find the derivative of each function.

a. $f(x) = e^{-\sqrt{x}}$

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b. $y = e^{\ln x^2}$

c. $g(t) = \left(e^{-2t} - e^t\right)^3$

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d. $f(x) = xe^x$

e. $y = \dfrac{e^{-x} - e^x}{e^{-x} + e^x}$

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- Derivatives for Bases Other than *e*

Theorem: Derivative for bases other than e

Let *a* be a positive real number and let u be a differentiable function of x.

1. $$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$$
2. $$\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \frac{du}{dx}$$
3. $$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$$
4. $$\frac{d}{dx}\log_a u = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

f. $y = \dfrac{3^{2x}}{x}$

$$y' = \frac{\frac{d}{dx}(3^{2x})x - 3^{2x}\left(\frac{d}{dx}x\right)}{x^2}$$

$$y' = \frac{(\ln 3)(3^{2x})(2)x - 3^{2x}}{x^2}$$

$$y' = \frac{3^{2x}(2x\ln 3 - 1)}{x^2}$$

or

$$y' = \frac{3^{2x}(\ln 3^{2x} - 1)}{x^2}$$

g. $\frac{d}{d\theta} g(\theta) = \frac{d}{d\theta}\left(4^{\theta}\sin(\pi\theta)\right)$

$$g'(\theta) = \left(\frac{d}{d\theta}4^{\theta}\right)\sin\pi\theta + 4^{\theta}\left(\frac{d}{d\theta}\sin\pi\theta\right)$$

$$g'(\theta) = (\ln 4)4^{\theta}\sin\pi\theta + 4^{\theta}(\pi\cos\pi\theta)$$

$$g'(\theta) = 4^{\theta}\left[(\ln 4)\sin\pi\theta + \pi\cos\pi\theta\right]$$

h. $\frac{d}{dx} y = \frac{d}{dx}\left(\dfrac{\log_4 x^{10}}{e^{5x^2}}\right) = 10\left[\frac{d}{dx}\left(\frac{\log_4 x}{e^{5x^2}}\right)\right]$

$$y' = 10\left[\frac{\left[\frac{d}{dx}(\log_4 x)\right]e^{5x^2} - (\log_4 x)\left[\frac{d}{dx}e^{5x^2}\right]}{\left(e^{5x^2}\right)^2}\right] \quad \left\{\frac{1}{\ln(\text{base})}\cdot\frac{u'}{u}\right\}$$

$$y' = 10\left[\frac{\frac{1}{(\ln 4)x}\cdot e^{5x^2} - \log_4 x\left[e^{5x^2}\cdot(10x)\right]}{\left(e^{5x^2}\right)^{2}}\cdot\frac{(\ln 4)x}{(\ln 4)x}\right] \rightarrow y' = 10\left(\frac{1-(10x^2)\ln 4}{x(\ln 4)e^{5x^2}}\right)$$

i. $r(s) = \sqrt[6]{s^5}\,\log_3\sqrt{1-s}$

- INTEGRALS OF EXPONENTIAL FUNCTIONS

Theorem: Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x dx = e^x + C$

2. $\int e^u du = e^u + C$

5. Integrate.

a. $\int \frac{2e^x - 2e^{-x}}{\left(e^x + e^{-x}\right)^2} dx$

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b. $\int \frac{e^{x^{-2}}}{x^3} dx$

Definition of Logarithmic Function to Base a

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base** a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x = \frac{\ln x}{\ln a}$$

Definition of Exponential Function the Base a

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base** a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}$$

$$a^x = e^{\ln a^x}$$

$$a^x = e^{x \ln a}$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

- Integrating
  - Option 1: Convert to base *e* using the formula $a^x = e^{(\ln a)x}$ and then integrate or
  - Option 2: Integrate directly using the integration formula

$$\int a^x dx = \left(\frac{1}{\ln a}\right) a^x + C$$

c. $\int 5^x dx = \int e^{(\ln 5)x} dx$

$u = (\ln 5)x$

$\frac{du}{dx} = \ln 5$

$dx = \frac{du}{\ln 5}$

$= \int e^u \left(\frac{du}{\ln 5}\right)$

$= \frac{1}{\ln 5} \int e^u du$

$= \frac{1}{\ln 5} e^u + C$

$= \frac{1}{\ln 5} e^{(\ln 5)x} + C$

$= \boxed{\frac{1}{\ln 5} 5^x + C}$

d. $\int \frac{3^{2x}}{1+3^{2x}} dx = \int (3^{2x})(1+3^{2x})^{-1} dx$

$u = 1 + 3^{2x}$

$\frac{du}{dx} = (\ln 3)3^{2x}(2)$

$dx = \frac{du}{(2\ln 3)3^{2x}}$

$= \int 3^{2x} u^{-1} \frac{du}{(2\ln 3)3^{2x}}$

$= \frac{1}{2\ln 3} \int u^{-1} du$

$= \frac{1}{2\ln 3} \ln|u| + C$

$= \boxed{\frac{1}{2\ln 3} \ln(1+3^{2x}) + C}$

e. $\int x^2 8^{-x^3} dx = \int x^2 8^u \left(-\frac{du}{3x^2}\right)$

$u = -x^3$

$\frac{du}{dx} = -3x^2$

$dx = -\frac{du}{3x^2}$

$= -\frac{1}{3} \int 8^u du$

$= -\frac{1}{3} \cdot \frac{8^u}{\ln 8} + C$

$= \boxed{\frac{-8^{-x^3}}{3\ln 8} + C}$

$\int x^2 e^{(\ln 8)(-x^3)} dx$

$u = (-\ln 8)x^3$

$\frac{du}{dx} = (-3\ln 8)x^2$

$dx = -\frac{du}{(3\ln 8)x^2}$

$= \int x^2 e^u \left(-\frac{du}{(3\ln 8)x^2}\right)$

$= -\frac{1}{3\ln 8} \int e^u du$

$= -\frac{1}{3\ln 8} e^u + C$

$= -\frac{1}{3\ln 8} e^{(\ln 8)(-x^3)} + C$

$= -\frac{1}{3\ln 8} 8^{-x^3} + C$

6. Find the area of the region bounded by the graphs of the equations $y = e^{-2x} + 2,\ y = 0,\ x = 0,$ and $x = 2.$

Note: $e^{-2x} + 2 > 0$

$A = \int_0^2 (e^{-2x} + 2)\,dx$

$A = \int_0^2 e^{-2x}\,dx + \int_0^2 2\,dx$

$u = -2x$

$\frac{du}{dx} = -2$

$dx = -\frac{du}{2}$

$A = \int_0^{-4} e^{u}\left(-\frac{du}{2}\right) + 2x\Big|_0^2$

upper: $u(2) = -2(2) = -4$

lower: $u(0) = -2(0) = 0$

$A = -\frac{1}{2}\int_0^{-4} e^{u}\,du + 2(2-0)$

$A = -\frac{1}{2}e^{u}\Big|_0^{-4} + 4$

$A = -\frac{1}{2}(e^{-4} - e^{0}) + 4$

$A = -\frac{1}{2}\left(\frac{1}{e^4} - 1\right) + 4$ sq. units

$A = -\frac{1}{2}\left(\frac{1-e^4}{e^4}\right) + 4\left(\frac{2e^4}{2e^4}\right)$

$A = \frac{8e^4 - (1 - e^4)}{2e^4}$

$A = \frac{9e^4 - 1}{2e^4}$ sq. units

7. Find an equation for the tangent line to the graph of $y = \log(2x)$ at the point $(5,1)$.

Find slope of y at (5,1)

$\frac{d}{dx}y = \frac{d}{dx}\log 2x$

$y' = \frac{2}{2x} \cdot \frac{1}{\ln 10}$

$y' = \frac{1}{x \ln 10}$

$y'(5) = \frac{1}{5\ln 10}$ = slope at (5,1)

Write equation of line tangent to y at (5,1)

$y - 1 = \frac{1}{5\ln 10}(x - 5)$