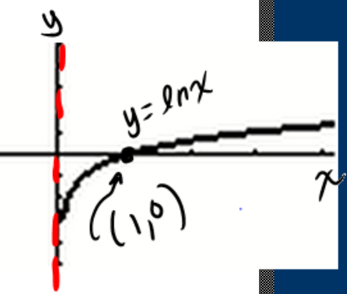
MATH 150/GRACEY NAME_____________________________

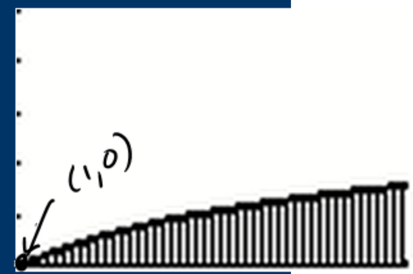
WORKSHEET/5.1

DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION



The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$



The domain of the natural logarithmic function is the set of all **positive** real numbers, $(0,\infty)$.

THEOREM: LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true:

The power to which I raise e to get 1 is zero

$2^a \cdot 2^b = 2^{a+b}$

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(\underbrace{a \cdot a \cdot a \cdots a}_{n\text{ times}})$
$= \ln a + \ln a + \cdots + \ln a$ n time
$= n \ln a$

$\frac{2^a}{2^b} = 2^{a-b}$

1. Sketch the graph of the function and state its domain and range.

$$f(x) = \ln(x-1)$$

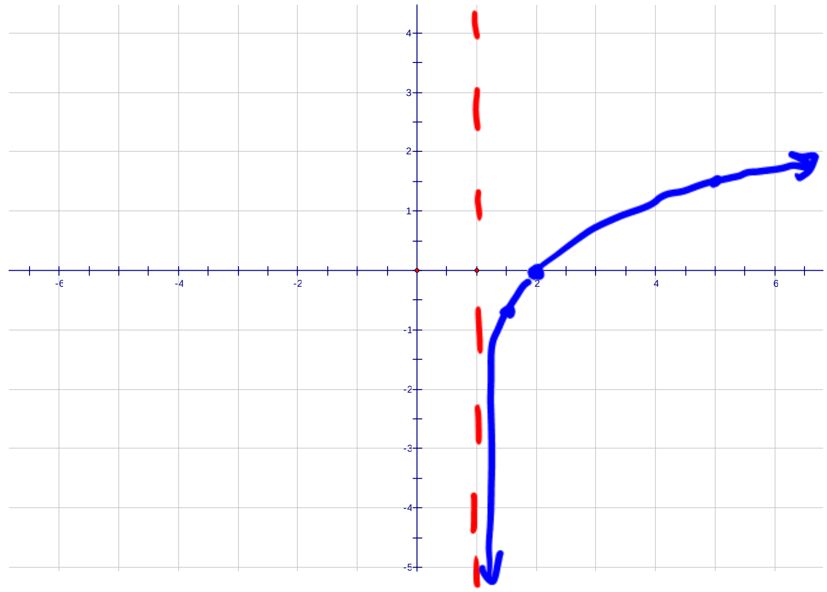
Domain: $(1, \infty)$

Range: $(-\infty, \infty)$

$f(2) = \ln(2-1)$
$= \ln 1$
$= 0$

$f(1.5) = \ln(1.5-1)$
$= \ln(.5)$
≈ -0.7

$f(5) = \ln 4$
$= 1.4$



2. Use the properties of logarithms to expand the logarithmic expression.

a. $\ln \dfrac{\sqrt[5]{x}}{y^2} = \ln x^{1/5} - \ln y^2$

$= \boxed{\frac{1}{5}\ln x - 2\ln y}$

$\ln(10)\cdot x \rightarrow x\ln 10$
$\ln 10x \rightarrow \ln(10x)$
$\ln 6 + 3 \rightarrow \ln(6) + 3$
$\ln(6+3) \rightarrow \ln 9$

b. $\ln\left(6e^3\right) = \ln 6 + \ln e^3$

$= \boxed{\ln 6 + 3}$

3. Write the expression as a logarithm of a single quantity.

a. $\ln(x+4)+\ln(x-4)$ $= \boxed{\ln(x+4)(x-4)}$

$= \boxed{\ln(x^2-16)}$

note: $(x+4)^2 = (x+4)\cdot(x+4)$

b. $\frac{1}{2}\left[3\ln x-\left(5\ln\left(x^3+2\right)+\ln x\right)\right]$

$= \frac{1}{2}\left[\ln x^3 - (\ln(x^3+2)^5 + \ln x\right]$

$= \frac{1}{2}\left[\ln x^3 - (\ln x(x^3+2)^5)\right]$

$= \frac{1}{2}\left[\ln \frac{x^3}{x(x^3+2)^5}\right]$

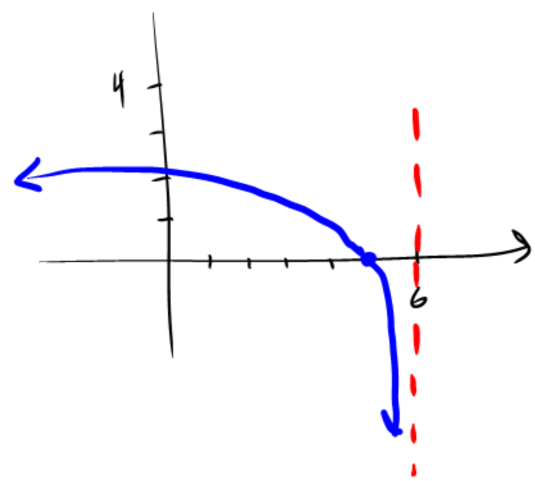
$= \boxed{\ln\sqrt{\frac{x^2}{(x^3+2)^5}}}$

$= \ln\left[\frac{|x|}{(x^3+2)^2\sqrt{x^3+2}}\right]$

4. Find the limit.

a. $\lim_{x\to 6^-}\ln(6-x) = \lim_{x\to 6^-}\ln[-(x-6)]$

$= -\infty \to \boxed{DNE}$



b. $\lim_{x\to 5^+}\ln\frac{x}{\sqrt{x-4}} \overset{D.S.}{=} \ln\frac{5}{\sqrt{5-4}}$

$= \boxed{\ln 5}$

THEOREM: DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x

$$\frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$$

or $\frac{d}{dx}\ln|u| = \frac{1}{u}\cdot\frac{du}{dx}$

5. Find an equation of the tangent line to the graph of the logarithmic function $y = \ln x^{1/2}$ at the point $(1, 0)$.

$\frac{d}{dx}y = \frac{d}{dx}\frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}\ln x$

$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x}\Big|_{x=1}$

$\frac{dy}{dx} = \frac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{2}(x - 1)$

$\boxed{y = \frac{1}{2}(x-1)}$

$\frac{d}{dx}y = \frac{d}{dx}\ln x^{1/2}$

$\frac{dy}{dx} = \frac{1}{x^{1/2}}\cdot\frac{d}{dx}(x^{1/2})$

$\frac{dy}{dx} = \frac{1}{x^{1/2}}\cdot\frac{1}{2x^{1/2}}$

$\frac{dy}{dx} = \frac{1}{2x}$

6. Find the derivative of the function.

a. $\frac{d}{dx} y = \frac{d}{dx} \ln\left(3x^4 - 5\right)$

$$\frac{dy}{dx} = \frac{1}{3x^4 - 5} \cdot \frac{d}{dx}\left(3x^4 - 5\right)$$

$$\boxed{\frac{dy}{dx} = \frac{12x^3}{3x^4 - 5}}$$

b. $\frac{d}{dx} y = \frac{d}{dx} x \ln x$

$$\frac{dy}{dx} = \left(\frac{d}{dx} x\right) \ln x + x\left(\frac{d}{dx} \ln x\right)$$

$$\frac{dy}{dx} = 1 \ln x + x\left(\frac{1}{x}\right)$$

$$\boxed{\frac{dy}{dx} = 1 + \ln x}$$

c. $f(x) = \ln\left(\frac{6x}{6x-5}\right)$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \ln 6x - \frac{d}{dx} \ln(6x-5)$$

$$f'(x) = \frac{1}{6x} \cdot \frac{d}{dx} 6x - \frac{1}{6x-5} \cdot \frac{d}{dx}(6x-5)$$

$$f'(x) = \frac{\cancel{6}^{1}}{\cancel{6}_{1} x} - \frac{6}{6x-5}$$

$$f'(x) = \frac{1(6x-5) - 6(x)}{x(6x-5)}$$

$$\boxed{f'(x) = -\frac{5}{x(6x-5)}}$$

d. $h(t) = \sqrt[4]{\frac{x-2}{x+2}}$

$$y = \ln \frac{6x}{6x-5}$$

$$y' = \frac{1}{\left(\frac{6x}{6x-5}\right)} \cdot \left[\frac{6(6x-5) - 6x(6)}{(6x-5)^2}\right]$$

$$y' = \frac{\cancel{6x-5}^{\,1}}{6x} \cdot \frac{36x-30-36x}{(6x-5)^{\cancel{2}\,1}}$$

$$y' = \frac{-\cancel{30}^{\,5}}{\cancel{6}_{1}x(6x-5)}$$

$$\boxed{y' = -\frac{5}{x(6x-5)}}$$

e. $y = \ln\sqrt{5+\sin^2 x}$

$$\frac{d}{dx} y = \frac{d}{dx}\frac{1}{2}\ln(5+\sin^2 x)$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\ln(5+\sin^2 x)\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{5+\sin^2 x}\cdot\frac{d}{dx}(5+\sin^2 x)$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{2\sin x\cos x}{5+\sin^2 x} = \underline{\frac{\sin 2x}{2(5+\sin^2 x)}} \text{ or } \frac{dy}{dx} = \underline{\frac{\sin x\cos x}{5+\sin^2 x}}$$

f. $\frac{d}{dx}\left(x^2 y - \ln(xy)\right) = \frac{d}{dx} 8y$, find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2 y) - \frac{d}{dx}(\ln xy) = 8\cdot\frac{dy}{dx}$$

$$\left(\frac{d}{dx}x^2\right)y + x^2\frac{d}{dx}y - \frac{1}{xy}\cdot\frac{d}{dx}(xy) = 8\frac{dy}{dx}$$

$$xy\left(2xy + x^2\cdot\frac{dy}{dx} - \frac{1\cdot y + x\cdot\frac{dy}{dx}}{xy}\right) = \left(8\frac{dy}{dx}\right)xy$$

$$2x^2y^2 + x^3y\frac{dy}{dx} - y - x\frac{dy}{dx} = 8xy\frac{dy}{dx}$$

$$x^3y\frac{dy}{dx} - 8xy\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x^2y^2$$

$$\frac{dy}{dx}\left[x(x^2y - 8y - 1)\right] = y(1-2x^2y)$$

$$\boxed{\frac{dy}{dx} = \frac{y(1-2x^2y)}{x(x^2y-8y-1)}}$$

7. Find the relative extrema and inflection points for the function $f(x) = \frac{\ln x}{x}$.

relative extrema

1) Find C.N.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\frac{\ln x}{x}$$

$$f'(x) = \frac{(1/x)x - (\ln x)(1)}{x^2}$$

$$f'(x) = \frac{1-\ln x}{x^2}$$

$$0 = 1 - \ln x$$

$$\ln x = 1$$

$$e^1 = x$$

$$x = e$$

$$c = e$$

2) Ran test for ↑, ↓ intervals

$(0,e)$ $+$ | (e,∞) $-$

0, 1, e, e^2

$$f'(x) = \frac{1-\ln x}{x^2}$$

$$f'(1) = \frac{1-0}{1^2} > 0$$

$$f'(e^2) = \frac{1-\ln e^2}{e^4} = \frac{-1}{e^4} < 0$$

3) run 1st derivative test

4) Conclusion

↑ to ↓ So there's a relative max at $(e, f(e)) = \left(e, \frac{1}{e}\right)$

Points of Inflection

① Find f' and f'', and find the zeros of f''

$$f(x) = \frac{\ln x}{x}$$

$$\frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{1-\ln x}{x^2}\right)$$

$$f''(x) = \frac{(-1/\cancel{x})\,x^{\cancel{2}} - (1-\ln x)(2x)}{x^4}$$

$$f''(x) = \frac{-\cancel{x}\,(1 + 2(1-\ln x))}{x^{\cancel{4}\,3}}$$

$$f''(x) = -\left(\frac{3 - 2\ln x}{x^3}\right)$$

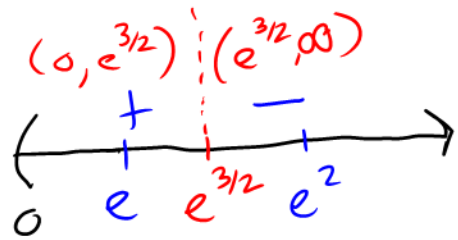
$$0 = \frac{3-2\ln x}{x^3}$$

$$2\ln x = 3$$

$$\ln x = \frac{3}{2}$$

$$e^{3/2} = x$$

② Run the test for concavity, and find out if theres any points of inflection



$$f''(x) = \frac{3-2\ln x}{x^3}$$

$$f''(e) = \frac{3-2(1)}{e^3} > 0$$

$$f''(e^2) = \frac{3-2(2)}{e^6} < 0$$

③ Conclusion

Since there's a change of concavity at $e^{3/2}$, there's a POI at

$$f(e^{3/2}) = \frac{\ln e^{3/2}}{e^{3/2}}$$

$$(e^{3/2}, f(e^{3/2})) = \boxed{\left(e^{3/2}, \frac{3}{2e^{3/2}}\right)}$$

8. Use logarithmic differentiation to find dy/dx.

$$\ln y = \ln\sqrt{(x-1)(x-2)(x-3)}$$

$$\boxed{\sqrt{(x-1)(x-2)(x-3)} = (x-1)^{1/2}(x-2)^{1/2}(x-3)^{1/2}}$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\frac{1}{2}\left[\ln(x-1) + \ln(x-2) + \ln(x-3)\right]$$

$$y\left(\frac{1}{y}\cdot\frac{dy}{dx}\right) = y\left(\frac{1}{2}\left[\frac{1\,(x-2)(x-3)}{x-1} + \frac{1\,(x-1)(x-3)}{x-2} + \frac{1\,(x-1)(x-2)}{x-3}\right]\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1(x-2)(x-3) + 1\cdot(x-1)(x-3) + 1(x-1)(x-2)}{(x-1)(x-2)(x-3)}\right]$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{x^2-5x+6+x^2-4x+3+x^2-3x+2}{(x-1)(x-2)(x-3)}\right]$$

$$\frac{dy}{dx} = \frac{(x-1)^{1/2}(x-2)^{1/2}(x-3)^{1/2}}{2}\cdot\frac{3x^2-12x+11}{(x-1)^1(x-2)^1(x-3)^1}$$

$+33$, -12

impossible

$$\frac{(x-1)^{1/2}}{(x-1)^1} = (x-1)^{1/2-1} = (x-1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{3x^2-12x+11}{2(x-1)^{1/2}(x-2)^{1/2}(x-3)^{1/2}}$$

$$\frac{dy}{dx} = \frac{3x^2-12x+11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

$f(t) = t^2+1$

$f(x) = x^2+1$

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

$\frac{x^3}{3} = \frac{1}{3}x^3$

Consider,

$$\int_2^x (t^2+1)\,dt = \left(\frac{t^3}{3} + t\right)\Big|_2^x$$

$$= \left(\frac{x^3}{3} + x\right) - \left(\frac{8}{3} + 2\right)$$

$$= x^3/3 + x - 14/3$$

$$\frac{d}{dx}\left(\frac{x^3}{3} + x - \frac{14}{3}\right)$$

$$= x^2 + 1$$

$$= f(x)$$

So $\frac{d}{dx}\left[\int_2^x (t^2+2)\,dt\right]$

$$= \boxed{x^2+2}$$

Consider...

$u=f(x)$

$$\frac{d}{dx} F(x) = \frac{d}{dx}\int_a^{u=f(x)} f(t)\,dt$$

$$F'(x) = \left[\frac{d}{du}\int_a^u f(t)\,dt\right]\frac{du}{dx}$$

$u = f(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Ex. 1

$$\frac{d}{dx} F(x) = \frac{d}{dx}\int_1^{x+3} \arcsin t\,dt$$

$$F'(x) = \underbrace{\frac{d}{du}\left[\int_1^u \arcsin t\,dt\right]}\frac{du}{dx}$$

$$\frac{d}{dx}u = \frac{d}{dx}(x+3)$$

$$\frac{du}{dx} = 1$$

$$F'(x) = (\arcsin u)(1)$$

$$\boxed{F'(x) = \arcsin(x+3)}$$

Ex. 2

$$\frac{d}{dx} F(x) = \frac{d}{dx}\int_{\tan x}^{5} \sqrt{t^3+4}\,dt$$

$$F'(x) = \frac{d}{du}\left[-\int_5^u \sqrt{t^3+4}\,dt\right]\frac{du}{dx}$$

$$\frac{d}{dx}u = \frac{d}{dx}\tan x$$

$$\frac{du}{dx} = \sec^2 x$$

$$F'(x) = \left(-\sqrt{u^3+4}\right)(\sec^2 x)$$

$$\boxed{F'(x) = (-\sec^2 x)\sqrt{\tan^3 x + 4}}$$