MATH 150/GRACEY NAME______________________

WORKSHEET/4.4

Theorem: The Fundamental Theorem of Calculus

> If a function $f$ is continuous on the closed interval $[a,b]$ and $F$ is an antiderivative of $f$ on the interval $[a,b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Guidelines for Using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of $f$, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

3. It is not necessary to include a constant of integration $C$ in the antiderivative because

$$\int_a^b f(x)\,dx = F(x) + C\Big]_a^b$$

$$= \left[F(b) + C\right] - \left[F(a) + C\right]$$

$$= F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \underbrace{F(x)}\Big|_a^b = F(b) - F(a)$$

1. Evaluate the definite integral.

$f(x) = 6$

$F(x) = 6x$

$a = -2$

$b = 6$

a. $\displaystyle\int_{-2}^{6} 6dx = \underbrace{6x}\Big|_{-2}^{6}$

$= 6(6) - 6(-2)$

$= \boxed{48}$

$f(x) = 2x^2 + 1$

$F(x) = \frac{2}{3}x^3 + x$

$a = 1$

$b = 6$

b. $\displaystyle\int_{1}^{6} \left(2x^2 + 1\right)dx = \left(\frac{2}{3}x^3 + x\right|_1^6$

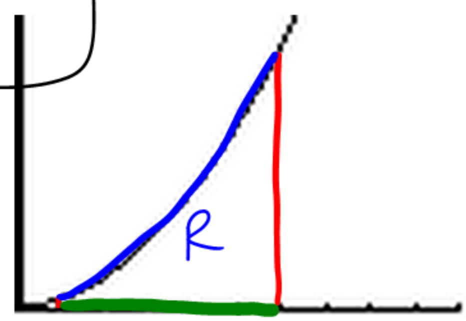
$= \left[\frac{2}{3}(6)^3 + (6)\right] - \left[\frac{2}{3}(1)^3 + (1)\right]$

$= \left[\frac{2}{\cancel{3}_1} \cdot \cancel{216}^{72} + 6\right] - \left[\frac{2}{3} + \frac{3}{3}\right]$

$= 150 - \frac{5}{3}$

$= \frac{450 - 5}{3}$

$= \boxed{\frac{445}{3}}$




$R = \displaystyle\int_{+1}^{6}(2x^2+1)dx$

c. $\displaystyle\int_{0}^{2} (2 - t)\sqrt{t}\,dt = \int_0^2 \left(2t^{1/2} - t^{3/2}\right)dt$

$= \left(\frac{2t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2}\right|_0^2$

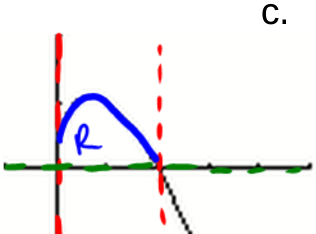
$= \left(\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right|_0^2$

$= \left[\frac{4}{3}\sqrt{2^3} - \frac{2}{5}\sqrt{2^5}\right] - \left[0 - 0\right]$

$= \frac{4}{3} \cdot 2\sqrt{2} - \frac{2}{5} \cdot 4\sqrt{2}$

$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$

$= \boxed{\frac{16}{15}\sqrt{2}}$




$R = \displaystyle\int_0^2 (2-t)\sqrt{t}\,dt$

d. $\displaystyle\int_{1}^{4} (2v + 5)^3 dv$

$(2v+5)^3 = 1(2v)^3(5)^0 + 3(2v)^2(5) + 3(2v)^1(5)^2 + 1(2v)^0(5)^3$

$= 8v^3 + 60v^2 + 150v + 125$

$\displaystyle\int (2v+5)^3 dv = \int_1^4 \left(8v^3 + 60v^2 + 150v + 125\right)dv$

$= 8\frac{v^4}{4} + 60\frac{v^3}{3} + 150\frac{v^2}{2} + 125v \Big|_1^4$

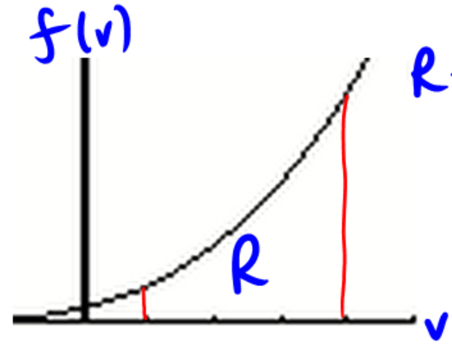
$$= 2v^4 + 20v^3 + 75v^2 + 125v \Big|_1^4$$

$$= \left[2(4)^4 + 20(4)^3 + 75(4)^2 + 125(4)\right] - \left[2 + 20 + 75 + 125\right]$$

$$= 3492 - 222$$

$$= \boxed{3270}$$

$$R = \int_1^4 (2v+5)^3\, dv$$




RATIONAL FUNCTIONS must be written as the sum/difference of functions (neg. exponents are ok)

Ex: $\displaystyle\int \frac{5x^3 - x^2 + 2}{x^2}\,dx = \int \left(5x - 1 + 2x^{-2}\right) dx$

power rule

Products of functions (not a constant times a function) must be multiplied out.

<u>Trig. functions</u>:

Identities → try changing everything into sines and cosines

Pythagorean Conjugates → you generate a difference of squares situation so that you can change the denominator from 2 terms to 1 term.

Ex: $\displaystyle\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{(1+\sin x)}\frac{(1-\sin x)}{(1-\sin x)}\,dx$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$\frac{1}{2+3} \overset{?}{=} \frac{1}{2} + \frac{1}{3}$$

$$\frac{1}{5} \neq \frac{5}{6}$$

False

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx$$

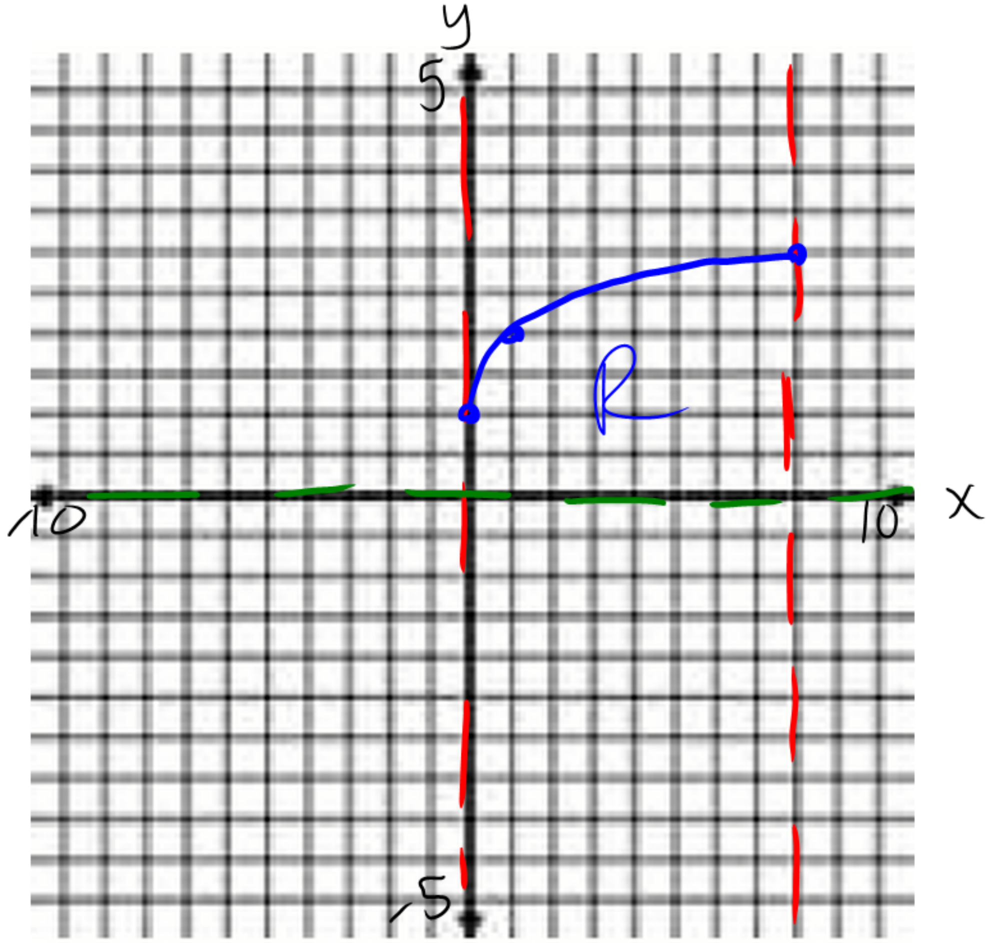
$$= \int \frac{1}{\cos^2 x}\,dx - \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\,dx$$

$$= \int \sec^2 x\,dx - \int \sec x \tan x\,dx$$

$$= \tan x - \sec x + C$$

2. Find the area of the region bounded by the graphs of the equations.

$$y = 1+\sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0.$$



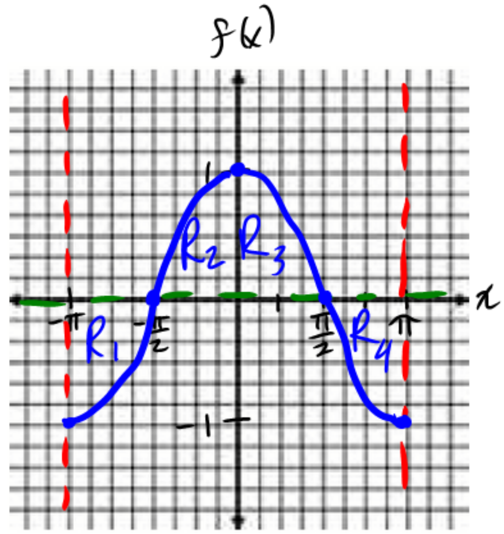

$$R = \int_0^8 (1+x^{1/3})\,dx$$

$$R = (x + \tfrac{3}{4}x^{4/3} \Big|_0^8$$

$$R = \left[8 + \tfrac{3}{4}(\sqrt[3]{8})^4\right] - \left[0 + 0\right]$$

$$R = 8 + \frac{3}{\cancel{4}} \cdot \overset{4}{\cancel{16}} \rightarrow \boxed{\text{Area} = 20 \text{ sq. units}}$$

Consider: $\int_{-\pi}^{\pi} \cos x \, dx$




$R_1 = R_4$ and $R_2 = R_3$

$$\int_{-\pi}^{\pi} \cos x \, dx = 2\int_{0}^{\pi} \cos x \, dx$$

$R_1$ and $R_4$ will have negative results

$$\int_{-\pi}^{\pi} \cos x \, dx = 2\int_{0}^{\pi} \cos x \, dx$$

$$= 2\sin x \Big|_{0}^{\pi}$$

$$= 2(0-0)$$

$$= \boxed{0}$$

What if we wanted to find the area?

$$A = 2\left[\int_{0}^{\pi/2} \cos x \, dx + \left|\int_{\pi/2}^{\pi} \cos x \, dx\right|\right]$$

$$A = 2\left[\sin x \Big|_{0}^{\pi/2} + |\sin x| \Big|_{\pi/2}^{\pi}\right]$$

$$A = 2\left[(1-0) + |0-1|\right]$$

$$A = 2 \cdot 2$$

$$\boxed{A = 4 \text{ sq. units}}$$

$$\int_{\pi/6}^{\pi/3} \frac{(1)(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1+\cos\theta}{1-\cos^2\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1+\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{\sin^2\theta}\,d\theta + \int_{\pi/6}^{\pi/3} \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta + \int_{\pi/6}^{\pi/3} \csc\theta\cot\theta\,d\theta$$

$$= -\cot\theta \Big|_{\pi/6}^{\pi/3} - \csc\theta \Big|_{\pi/6}^{\pi/3}$$

$$= -\left(\frac{\cos \pi/3}{\sin \pi/3} - \frac{\cos \pi/6}{\sin \pi/6}\right) - \left(\frac{1}{\sin \pi/3} - \frac{1}{\sin \pi/6}\right)$$

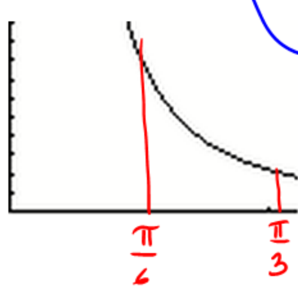
$$= -\left(\frac{1/\cancel{2}}{\sqrt{3}/\cancel{2}} - \frac{\sqrt{3}/\cancel{2}}{1/\cancel{2}}\right) - \left(\frac{1}{\sqrt{3}/2} - \frac{1}{1/2}\right)$$

$$= -\frac{1}{\sqrt{3}} + \sqrt{3} - \frac{2}{\sqrt{3}} + 2$$

$$= -\frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}} + \sqrt{3} + 2$$

$$= -\sqrt{3} + \sqrt{3} + 2$$

$$= \boxed{2}$$




$$\int_2^6 (x^2-8)^2\,dx = \int_2^6 (x^4 - 16x^2 + 64)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{16x^3}{3} + 64x \right|_2^6$$

$$= \left[\frac{(6)^5}{5} - \frac{16(6)^3}{3} + 64(6)\right] - \left[\frac{(2)^5}{5} - \frac{16(2)^3}{3} + 64(2)\right]$$

$$= \frac{3936}{5} - \frac{1376}{15}$$

$$= \boxed{\frac{10432}{15}}$$

$$\approx 695.5$$

```
fnInt((X²-8)²,X,
2,6)
         695.4666667
```

$$\int (x)^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$(x^2+2)^2 \to (\text{function } x)^2$$

$\int_{-a}^{a} f(x)\,dx$, f is even

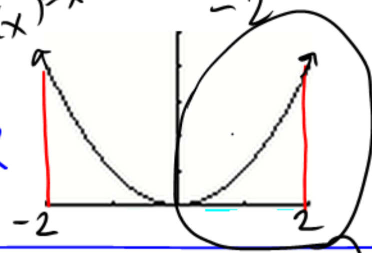
$\int_{-2}^{2} x^2\,dx = 2\int_{0}^{2} x^2\,dx$

$= 2\left(x^3/3 \right|_0^2$

$= 2\left(\frac{(2)^3}{3} - \frac{(0)^3}{3}\right)$

$= 16/3$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



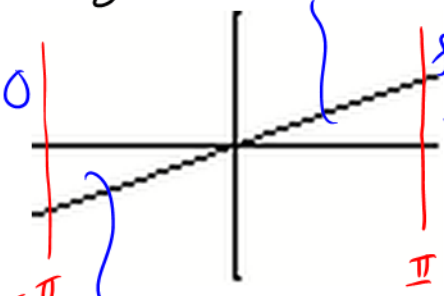

$2\int_{0}^{2} x^2\,dx$

$\int_{-a}^{a} f(x)\,dx$, f is odd

$$\int_{-a}^{a} f(x)\,dx = 0$$

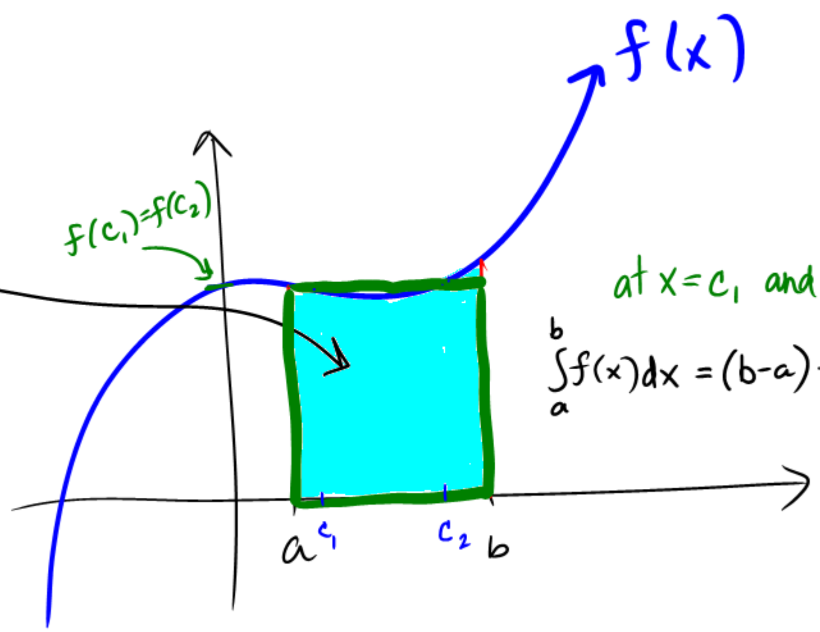
$\int_{-\pi/6}^{\pi/6} \sin x\,dx = 0$

$\int_{0}^{\pi/6} \sin x\,dx > 0$




$\int_{-\pi/6}^{0} \sin x\,dx < 0$

$\int_{a}^{b} f(x)\,dx$




at $x=c_1$ and at $x=c_2$

$\int_{a}^{b} f(x)\,dx = (b-a)\cdot f(c_1)$ average value of function

THE MEAN VALUE THEOREM FOR INTEGRALS

If $f$ is continuous on the closed interval $[a,b]$, then there exists a number $c$ in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

area under curve = area of rectangle

Definition of the Average Value of a Function on an Interval

$$\int_a^b f(x)dx=(b-a)f(c)$$

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

If $f$ is integrable on the closed interval $[a,b]$, then the **average value** of $f$ on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

3. Find the value(s) of $c$ guaranteed by the Mean Value Theorem for Integrals for the function $f(x)=\cos x$, $\left[-\frac{\pi}{3},\frac{\pi}{3}\right]$

$$\int_{-\pi/3}^{\pi/3} \cos x\,dx = f(x)\left[\frac{\pi}{3}-\left(-\frac{\pi}{3}\right)\right]$$

$$2\int_0^{\pi/3}\cos x\,dx = \frac{2\pi}{3}\cos x$$

$$2\left[\sin x\,\Big|_0^{\pi/3}\right] = \frac{2\pi}{3}\cos x$$

$$\sin\frac{\pi}{3}-\sin 0 = \frac{\pi}{3}\cos x$$

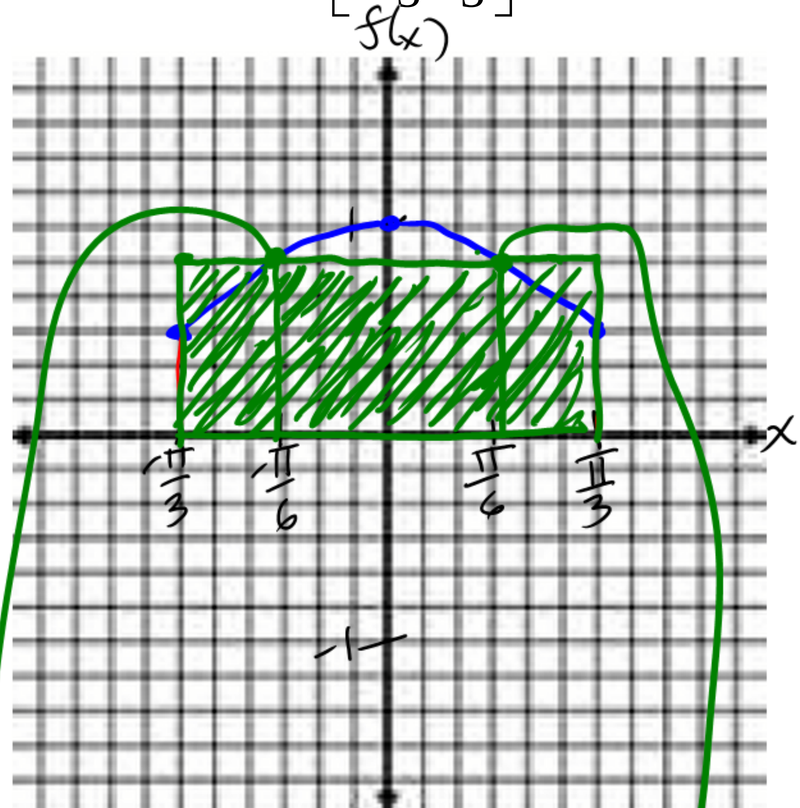
$$\frac{\sqrt{3}}{2}-0 = \frac{\pi}{3}\cos x$$

$$0.83 \approx \frac{3\sqrt{3}}{2\pi} = \cos x$$

average value

$x \approx \pm 0.5971$

$c \approx \pm 0.5971$




$(c_2, f(c_2))$

$(-0.5971, 0.83)$

$(c_1, f(c_1))$

$(0.5971, 0.83)$

4. Find the average value of the function $f(x) = \frac{4(x^2+1)}{x^2}$, $[1,3]$ and all the values of $x$ in the interval for which the function equals its average value.

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(c) = \frac{1}{3-1}\int_1^3 4(1+x^{-2})\,dx$$

$$f(c) = \frac{4}{2}\int_1^3 (1+x^{-2})\,dx$$

$$f(c) = 2\left(x + \frac{x^{-1}}{-1}\right|_1^3$$

$$f(c) = 2\left(x - \frac{1}{x}\right|_1^3$$

$$f(c) = 2\left[\left(3-\tfrac{1}{3}\right) - \left(1 - \tfrac{1}{1}\right)\right]$$

$$f(c) = 2\left[\tfrac{8}{3} - 0\right]$$

$$\boxed{f(c) = \tfrac{16}{3} = \text{average value}}$$

$$f(x) = 4(1+x^{-2})$$

$$f(c) = 4\left(1+\frac{1}{c^2}\right)$$

$b=3$

$a=1$