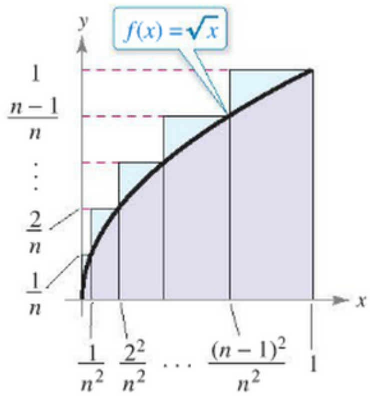
4.3



Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$[x_{i-1}, x_i]$. *ith subinterval*

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\, \Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ. (The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.)

$\|\Delta\|$ is called the norm of delta → the width of the largest subinterval

$$\|\Delta\| = \Delta x = \frac{b-a}{n}.$$ Regular partition

$$\frac{b-a}{\|\Delta\|} \le n$$ General partition

as $\|\Delta\| \to 0$,

$n \to \infty$

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\, dx$ exists.

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\, dx.$$

(See Figure 4.22.)

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

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WORKSHEET / 4.3

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \cancel{\frac{n^2+n}{2}}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \cancel{\frac{2n^3+3n^2+n}{6}}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \cancel{\frac{n^4+2n^3+n^2}{4}}$$

1. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$ over the region bounded by the graphs of $f(x)=\sqrt[3]{x},\ y=0,\ x=0,\ x=1.$ Hint: Let $c_i = i^3/n^3$ and recall that the width of each interval is $\Delta x_i = \frac{i^3}{n^3}-\frac{(i-1)^3}{n^3}$.

$$A^3-B^3=(A-B)(A^2+AB+B^2)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\cdot\left(\frac{i^3}{n^3}-\frac{(i-1)^3}{n^3}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\cdot\frac{1}{n^3}\right)\left[(i-(i-1))\left((i)^2+(i)(i-1)+(i-1)^2\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left[i(-1)(i^2+i^2-i+i^2-2i+1)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(-i)(3i^2-3i+1)$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\left[-3\sum_{i=1}^{n} i^3+3\sum_{i=1}^{n} i^2-\sum_{i=1}^{n} i\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\left[-3\frac{n^2(n+1)^2}{4}+\cancel{3}\frac{n(n+1)(2n+1)}{\cancel{6}\,2}-\frac{n(n+1)}{2}\right]$$

$$=\lim_{n\to\infty}\left(-\frac{3(n+1)^2}{4n^2}+\frac{(n+1)(2n+1)}{2n^3}-\frac{n+1}{2n^3}\right)$$

$$=\boxed{-\frac{3}{4}}$$

2. Evaluate the definite integral by the limit definition.

$$\int_1^6 \left(2x^2+1\right)dx$$

$$= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{20i}{n}+\frac{50i^2}{n^2}\right)\left(\frac{5}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\sum_{i=1}^{n} 15 + \frac{100}{n^2}\sum_{i=1}^{n} i + \frac{250}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{\cancel{n}}\cdot 15\cancel{n} + \frac{\overset{50}{\cancel{100}}}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)}{\underset{1}{\cancel{2}}} + \frac{\overset{125}{\cancel{250}}}{n^{\cancel{3}2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{\underset{3}{\cancel{6}}}\right]$$

$$= 15 + 50 + \frac{125}{3}$$

$$= \boxed{\frac{320}{3}}$$

$a=1, b=6, \Delta x_i = \Delta x = \frac{6-1}{n} = \frac{5}{n}$

$c_i: \ a+i\Delta x = 1+\frac{5i}{n}$

$f(c_i) = 2\left(1+\frac{5i}{n}\right)^2+1$

$= 2\left(1+2\cdot 1\cdot\frac{5i}{n}+\frac{25i^2}{n^2}\right)+1$

$= 2+\frac{20i}{n}+\frac{50i^2}{n^2}+1$

$= 3+\frac{20i}{n}+\frac{50i^2}{n^2}$

3. Write the limit as a definite integral on the interval $[a,b]$ where c_i is any point on the ith interval.

a. $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(8c_i+15)\Delta x_i, \quad [2,6]$

$$= \int_2^6 (8x+15)\,dx$$

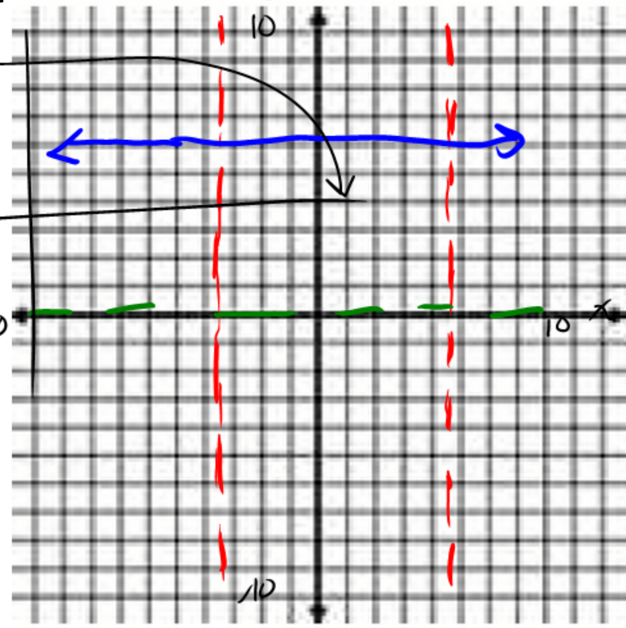
b. $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}5c_i\sqrt{c_i^2+2}\,\Delta x_i, \quad [0,12]$

$$= \int_0^{12} 5x\sqrt{x^2+2}\,dx$$

4. Sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

a. $\int_{-2}^{6} 6\,dx$

$y = 6$



$b \cdot h$

Area$_{\square}$ $= (6-(-2))(6)$

$= 48$ sq. units

b. $\int_{-4}^{4} \sqrt{16-x^2}\,dx$

$(y)^2 = (\sqrt{16-x^2})^2$

$y^2 = 16 - x^2$

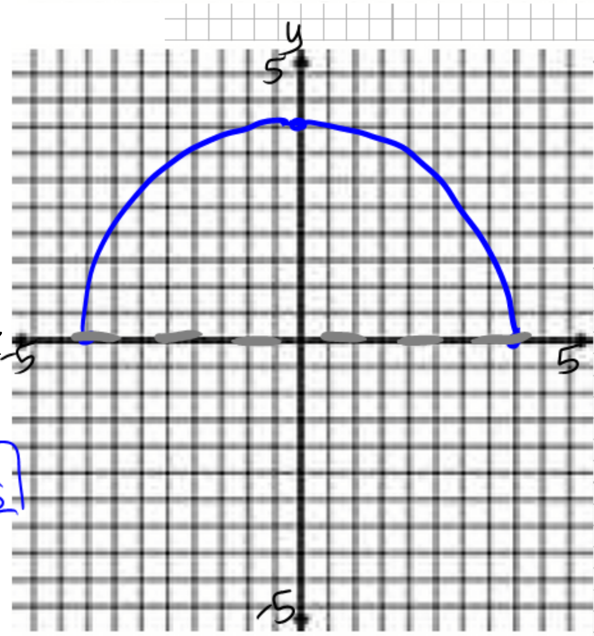
$x^2 + y^2 = 4^2$

upper half

$A = \frac{\pi r^2}{2}$

$= \frac{\pi (4)^2}{2}$

$= \boxed{8\pi \text{ sq. units}}$



5. Given $\int_0^3 f(x)dx = 4$ and $\int_3^6 f(x)dx = -1,$ evaluate

a. $\int_0^6 f(x)dx = \int_0^3 f(x)dx + \int_3^6 f(x)dx$

$= 4 + (-1) = \boxed{3}$

b. $\int_6^3 f(x)dx = -\int_3^6 f(x)dx$

$= -(-1) = \boxed{1}$

c. $\int_3^3 f(x)dx = \boxed{0}$

d. $\int_3^6 f(x)dx = \boxed{-1}$