When you finish your homework you should be able to…

- π Write the general solution of a differential equation
- π Use indefinite integral notation for antiderivatives
- π Use basic integration rules to find antiderivatives
- π Find a particular solution of a differential equation

Warm-up: For each derivative, describe the original function F.

1. $F'(x) = 2x$

$F(x) = x^2$

or $F(x) = x^2 - 1000$

2. $F'(x) = x^3$

$F(x) = \frac{1}{4} 4x^4 = \frac{1}{4}x^4$

or $F(x) = \frac{1}{4}x^4 + 4$

3. $F'(x) = \frac{1}{x^2} = x^{-2}$

$F(x) = --x^{-1}$

$F(x) = -x^{-1}$ or $F(x) = \frac{-1}{x} + 1$

4. $F'(x) = \sec^2 x$

$F(x) = \tan x$

or $F(x) = (\tan x) + 60$

5. $F'(x) = \sin x$

$F(x) = --\cos x = --\cos x$

or $F(x) = (-\cos x) - \pi$

6. $F'(x) = 6$

$F(x) = 6x$

or $F(x) = 6x + 6$

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Why is F called **an** antiderivative of f, rather than **the** antiderivative of f?

There are infinitely many antiderivative when you consider a constant term that could have ~~zeroed~~ out during differentiation.

THEOREM: REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x)=F(x)+C$, for all x in I where C is a constant.

NOTATION:

$$\int \overbrace{f(x)}^{\text{function to integrate — integrand}}\, dx = \underbrace{F(x)}_{\text{an antiderivative}} + \overbrace{C}^{\text{constant of integration}} = G(x)$$

integrate with respect to x

note: f(x) is the F′(x) in the warm-up

Example 1: Verify the statement by showing that the derivative of the right side equals the integrand of the left side.

$$\int\left(8x^3+\frac{1}{2x^2}\right)dx = 2x^4-\frac{1}{2x}+C$$

$$8x^3+\frac{1}{2x^2} \overset{?}{=} \frac{d}{dx}\left(2x^4-\frac{1}{2x}+C\right)$$

$$8x^3+\frac{1}{2x^2} \overset{?}{=} 8x^3-\frac{1}{2}\left(-x^{-2}\right)+0$$

$$8x^3+\frac{1}{2x^2} = 8x^3+\frac{1}{2x^2} \checkmark$$

Example 2: Find the general solution of the differential equation.

a. $\cancel{dx}\,\dfrac{dy}{\cancel{dx}}=2x^{-3}\,dx$

$$\int 1\,dy = \int 2x^{-3}\,dx$$

$$y + C_1 = \frac{2x^{-2}}{-2} + C_2$$

$$y = -x^{-2} + C_2 - C_1$$

$$\boxed{y = -x^{-2} + C}$$

b. $d\theta\,\dfrac{dr}{d\theta}=\pi\,d\theta$

$$\int 1\,dr = \int \pi\,d\theta$$

$$\boxed{r = \pi\theta + C}$$

BASIC INTEGRATION RULES

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C]=0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)]=kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx = kF(x)+C$
$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$	$\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \to F(x)\pm G(x)+C$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C,\ n\neq -1$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x]=-\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$

Example 3: Find the indefinite integral and check the result by differentiation.

a. $\int (16 - x)\,dx$

$= \int 16\,dx - \int x\,dx$

$= \boxed{16x - \frac{1}{2}x^2 + C}$

b. $\int \frac{\sqrt[5]{x^3} - 2x}{\sqrt{x}}\,dx$ quotient

$= \int \left(\frac{x^{3/5}}{x^{1/2}} - \frac{2x}{x^{1/2}} \right) dx$

$= \int \left(x^{1/10} - 2x^{1/2} \right) dx$ diff of 2 functions > power rule

$= \frac{x^{\frac{1}{10}+1}}{\frac{1}{10}+1} - \frac{2x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \boxed{\frac{10}{11}x^{11/10} - \frac{4}{3}x^{3/2} + C}$

c. $\int (3x-4)^3\,dx$

$\int \left(1\cdot(3x)^3(-4)^0 + 3(3x)^2(-4)^1 + 3(3x)^1(-4)^2 + 1\cdot(3x)^0(-4)^3 \right) dx$

$= \int \left(27x^3 - 108x^2 + 144x - 64 \right) dx$

$= 27\frac{x^{3+1}}{3+1} - 108\frac{x^{2+1}}{2+1} + 144\frac{x^{1+1}}{1+1} - 64x + C$

$= \boxed{\frac{27}{4}x^4 - 36x^3 + 72x^2 - 64x + C}$

d. $\int (1-u)\sqrt{u}\,du$ product 2 functions

$= \int \left(u^{1/2} - u^{3/2} \right) du$ diff of 2 functions

$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$

$= \boxed{\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C}$

e. $\int \sec t(\tan t - \sec t)\,dt$ prod. of 2 functions

$= \int \sec t \tan t\,dt - \int \sec^2 t\,dt$ diff of 2 trig integrals

$= \sec t - \tan t + C$

$\int \csc^2\theta\,d\theta = -\cot\theta$

f. $\int \left(4\theta - \csc^2\theta \right) d\theta$

$= \frac{4\theta^{1+1}}{1+1} - (-\cot\theta) + C$

$= \boxed{2\theta^2 + \cot\theta + C}$

Identities

g. $\int \frac{\sin x}{1-\sin^2 x}\,dx$

Know how to use Pythagorean conjugates
$1+\cos\theta$, $1-\cos\theta$
$1+\sin\theta$, $1-\sin\theta$

$= \int \frac{\sin x}{\cos^2 x}\,dx$

$= \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\,dx$

$= \int \sec x \tan x\,dx$

$= \boxed{\sec x + C}$

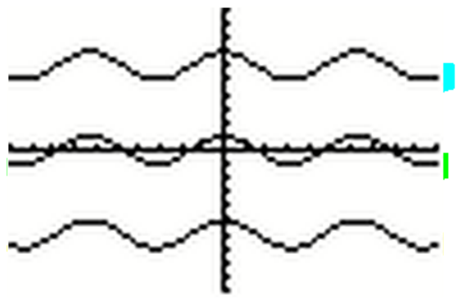
INITIAL CONDITIONS AND PARTICULAR SOLUTIONS

You have already seen that the equation $y=\int f(x)dx$ has __many__ solutions, each differing from each other by a __constant__.

This means that the graphs of any two __antiderivatives__ of f are __vertical__ translations of each other.

$\int \sin x\, dx = \cos x + C$

Family of antiderivatives



$F_1(x) = \cos x$

$F_2(x) = 6 + \cos x$

$F_3(x) = -6 + \cos x$

In many applications of integration, you are given enough information to determine a __particular__ solution. To do this, you need only know the value of $y=F(x)$ for one value of x. This information is called an __initial__ condition.

Example 4: Solve the differential equation.

a. $g'(x)=6x^2,\ g(0)=-1$

$\frac{dy}{dx}=6x^2$

$\int dy=\int 6x^2\,dx$

$y=\frac{6x^{2+1}}{2+1}+C$

$y=2x^3+C$

$g(x)=2x^3+C$ ← general solution

$-1=2(0)+C$

$-1=C$

$\boxed{g(x)=2x^3-1}$ ← particular solution

b. $f''(x)=\sin x,\ f'(0)=1,\ f(0)=6$

$\cancel{dx}\,\frac{d^2y}{dx^2}=\sin x\,dx$

$\int \frac{d}{dx}(dy)=\int \sin x\,dx$

$\frac{dy}{dx}=-\cos x+C$

$f'(x)=-\cos x+C$

$1=-\cos 0+C$

$1=-1+C$

$2=C$

$f'(x)=-\cos x+2$

$dx\,\frac{dy}{dx}=(-\cos x+2)\,dx$

$\int dy=\int(-\cos x+2)\,dx$

$y=-\sin x+2x+C$

$f(x)=-\sin x+2x+C$

$6=-\sin 0+2\cdot 0+C$

$6=C$

$\boxed{f(x)=-\sin x+2x+6}$