When you finish your homework you should be able to…

- π Determine finite limits at infinity
- π Determine the horizontal asymptotes, if any, of the graph of a function
- π Determine infinite limits at infinity

Warm-up: Evaluate the following limits analytically

1. $\lim_{x\to 1^+} \frac{3}{1-x} = -\infty$

2. $\frac{3}{3}\lim_{t\to 0} \frac{\sin 3t}{t} = 3 \lim_{t\to 0} \frac{\sin 3t}{3t}$

$= 3(1)$

$= \boxed{3}$

LIMITS AT INFINITY

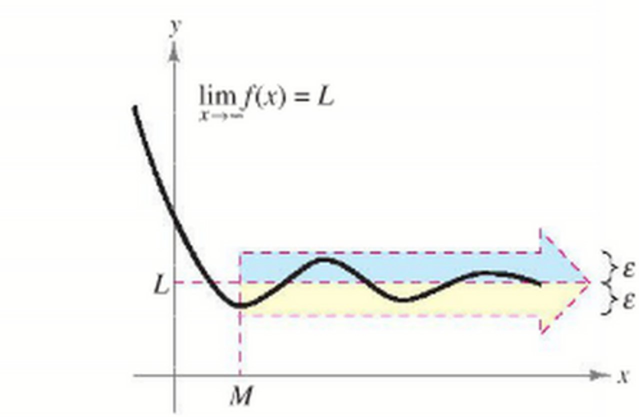
This section discusses the __end______ behavior of a function on an

__infinite______ interval.

DEFINITION OF LIMITS AT INFINITY

Let L be a real number.

1. The statement $\lim\limits_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim\limits_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$, such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



DEFINITION OF A HORIZONTAL ASYMPTOTE

The line $y = L$ is a **horizontal asymptote** of the graph of f if $\lim\limits_{x\to-\infty} f(x) = L$ or $\lim\limits_{x\to\infty} f(x) = L$.

THEOREM: LIMITS AT INFINITY

If r is a positive rational number, then

$$\lim_{x\to\infty}\frac{c}{x^r}=0.$$

Furthermore, if x^r is defined when $x<0$, then

$$\lim_{x\to-\infty}\frac{c}{x^r}=0.$$

calculator:

$\frac{2}{1000^3}$

$\approx 2.0\times10^{-9}$

$= 0.000000002$

$c=2$

$x=1000$

$r=3$

$\frac{2}{(-1000)^3}\approx -2\times10^{-9}$

-0.000000002

DEFINITION OF INFINITE LIMITS AT INFINITY

Let f be a function defined on the interval (a,∞).

1. The statement $\lim_{x\to\infty} f(x)=\infty$ means that for each $M>0$ there is a corresponding number $N>0$, such that $f(x)>M$ whenever $x>N$.
2. The statement $\lim_{x\to\infty} f(x)=-\infty$ means that for each $M<0$ there is a corresponding number $N>0$, such that $f(x)<M$ whenever $x>N$.

GUIDELINES FOR FINDING LIMITS AT +/- INFINITY

1. If the degree of the numerator is **less than** the degree of the denominator, then the limit of the rational function is ___0___.
2. If the degree of the numerator is **equal to** the degree of the denominator, then the limit of the rational function is the ___ratio___ of the leading ___coefficients___.
3. If the degree of the ~~denominator~~ numerator is **greater than** the degree of the denominator, then the limit of the rational function is plus or minus infinity, hence it does not ___exist___.

Example 1: Find the limit.

a. $\lim\limits_{x\to-\infty}\left(\frac{5}{x}-\frac{x}{3}\right)$

$$= \lim_{x\to-\infty}\frac{5}{x^1} - \lim_{x\to-\infty}\frac{x}{3}$$

$$= 0-(-\infty)$$

$= \infty \to$ DNE, but gives important info about end behavior of the function

b. $\lim\limits_{x\to\infty}\left(\frac{x^2+3}{2x^2-1}\right)$

$$= \lim_{x\to\infty}\frac{\frac{x^2+3}{x^2}}{\frac{2x^2-1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{1+\frac{3}{x^2}}{2-\frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{3}{x^2}}{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(1+\frac{3}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2-\frac{1}{x^2}\right)}$$

$$= \frac{1+0}{2-0}$$

$$= \boxed{\frac{1}{2}}$$

c. $\lim\limits_{x\to+\infty}\sqrt{\frac{x^4-1}{x^3-1}}$

$$= \sqrt{\lim_{x\to\infty}\frac{\frac{x^4-1}{x^3}}{\frac{x^3-1}{x^3}}}$$

$$= \sqrt{\lim_{x\to\infty}\frac{x-\frac{1}{x^3}}{1-\frac{1}{x^3}}}$$

$$= \sqrt{\frac{\lim\limits_{x\to\infty}x-\lim\limits_{x\to\infty}\frac{1}{x^3}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{1}{x^3}}}$$

$$= \sqrt{\frac{\infty-0}{1-0}}$$

$$= \sqrt{\infty}$$

$= \infty \to$ DNE

d. $\lim_{x\to\infty} \cos\frac{1}{x}$

$= \cos\left(\lim_{x\to\infty} \frac{1}{x}\right)$

$= \cos 0$

$= \boxed{1}$

e. $\lim_{x\to+\infty} \frac{-3x+1}{\sqrt{x^2+x}}$

$\sqrt{x^2} = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

$= \lim_{x\to\infty} \frac{\frac{-3x+1}{\sqrt{x^2}}}{\sqrt{\frac{x^2+x}{x^2}}}$

$= \frac{\lim_{x\to\infty} \frac{-3x+1}{x}}{\sqrt{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)}}$

$= \frac{\lim_{x\to\infty}\left(-3+\frac{1}{x}\right)}{\sqrt{1+0}}$

$= \frac{-3+0}{1}$

$= \boxed{-3}$

Note: since $x\to+\infty$, we did not need to consider $-x$

f. $\lim_{x\to\infty} \frac{x}{x^2-1}$

$= \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2-1}{x^2}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}}$

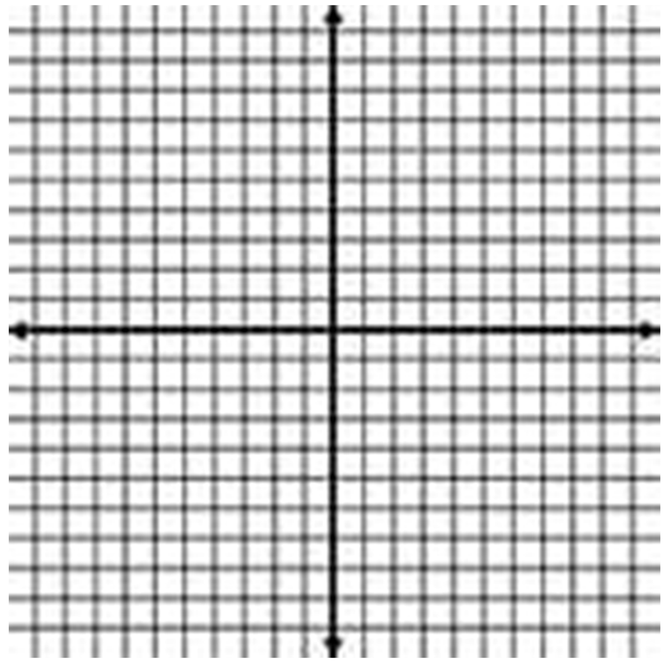
$= \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)}$

$= \frac{0}{1-0}$

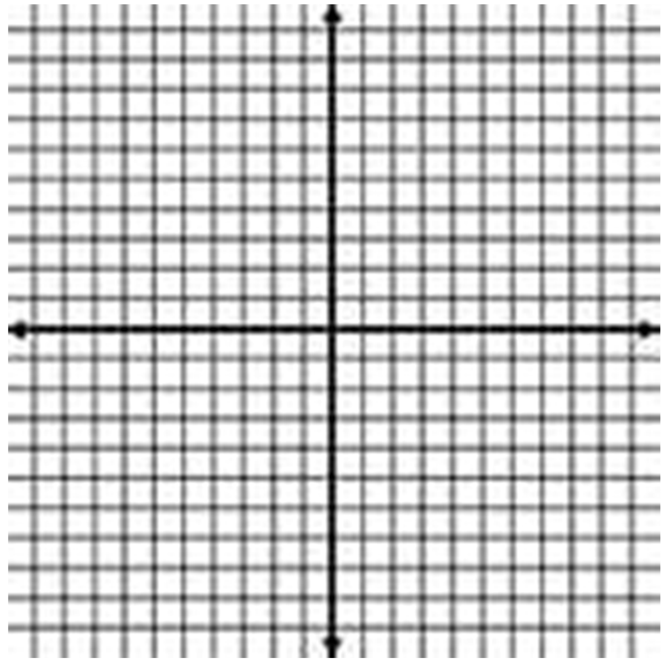
$= \boxed{0}$

Example 2: Sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.

a. $$f(x)=\frac{1}{x^2-x-2}$$



b. $h(x)=\dfrac{2x}{\sqrt{3x^2+1}}$



Intercepts:

x-int: $0 = 2x$
$x = 0$

y-int: $h(0)=\dfrac{2(0)}{\sqrt{3(0)^2+1}}$
$= 0$

Asymptotes:

Vertical: $\left(\sqrt{3x^2+1}\right)^2=(0)^2$
$3x^2+1=0$
$3x^2=-1$ not real

$\boxed{\text{NONE}}$

Horizontal:

$$\lim_{x\to\infty}\frac{2x}{\sqrt{3x^2+1}}=\lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{3x^2+1}{x^2}}}$$

$x>0$ since $x\to\infty$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{3+1/x^2}}$$

$$=\frac{+2}{\sqrt{3+0}}$$

$$=\frac{2}{\sqrt{3}}$$

$$\lim_{x\to-\infty}\frac{\frac{2x}{-x}}{\sqrt{3+1/x^2}}=\frac{-2}{\sqrt{3}}$$

$$y=\pm\frac{2}{\sqrt{3}}$$