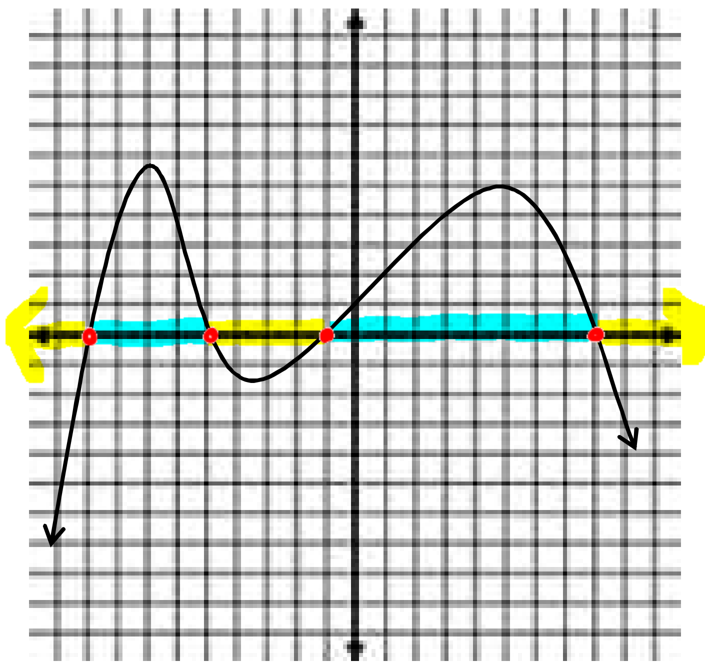
When you are done with your homework you should be able to…

- π Determine intervals on which a function is concave upward or concave downward
- π Find any points of inflection of the graph of a function
- π Apply the Second Derivative Test to find relative extrema of a function

Warm-up: Consider the graph of f' shown below.



a. Identify the interval(s) on which f is

i. increasing $\rightarrow f' > 0$

$(-9,-5) \cup (-1,8)$

ii. decreasing $\rightarrow f' < 0$

$(-\infty,-9) \cup (-5,-1) \cup (8,\infty)$

b. Estimate the value(s) of x at which f has a relative

i. minimum $x = -9, -1$

ii. maximum $x = -5, 8$

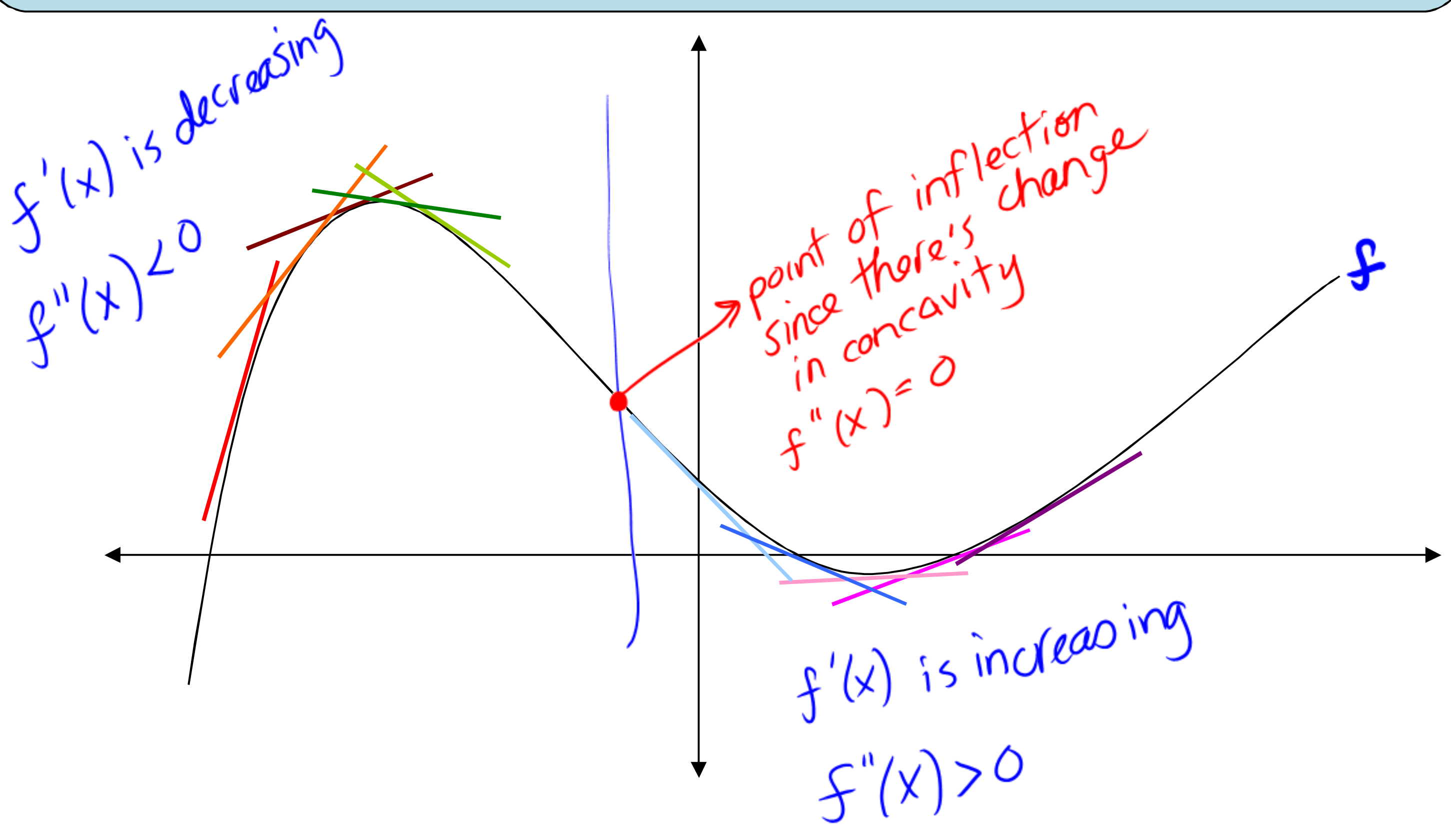
DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

THEOREM: TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is **concave upward** on I.
2. If $f''(x) < 0$ for all x in I, then f is **concave downward** on I.



Example 1: Identify the open intervals on which the function is concave upward or concave downward.

a. $y=-x^3+3x^2-2$

$y'(x) = -3x^2+6x$

$y''(x) = -6x+6$

$0 = -6x+6$

$6x=6$

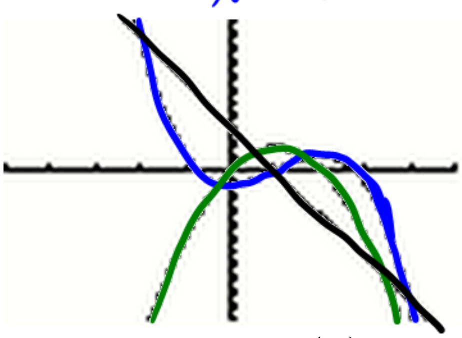
$x=1$

$y''(x) = -6(x-1)$

$y''(0) = -6(0-1) = 6 > 0$

$y''(2) = -6(2-1) = -6 < 0$

$\boxed{y \text{ is concave upward on } (-\infty,1) \text{ and concave downward on } (1,\infty)}$



$f(x) = -x^3+3x^2-2$

$f'(x) = -3x^2+6x$

$f''(x) = -6x+6$

b. $f(x)=x+\dfrac{2}{\sin x},\ [-\pi,\pi]$

$f(x) = x+2\csc x$

$f'(x) = 1-2\csc x\cot x$

$f''(x) = -2\left[(-\csc x\cot x)\cot x+\csc x(-\csc^2 x)\right]$

$f''(x) = 2\csc x(\cot^2 x+\csc^2 x)$

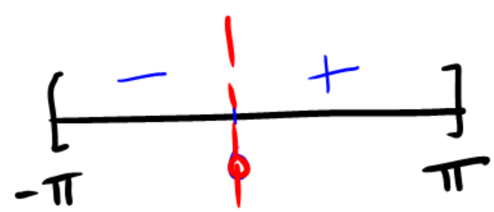
$\cancel{2\csc x} = 0$

$\cot^2 x+\csc^2 x = 0$

$\dfrac{\cos^2 x+1}{\sin^2 x} = 0$

$\cos^2 x = -1$

no real zeros



$f''(-\frac{\pi}{2}) = 2(-1)(0+1)$

$f''(\frac{\pi}{2}) = 2(1)(0+1)$

f is concave upward on $(0,\pi)$ and concave downward on $(-\pi,0)$

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be an element in the interval. If the graph of f has a tangent line at the point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward or from downward to upward at the point.

THEOREM: POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c)=0$ or f'' does not exist at $x=c$. Points of inflection only if there's a change in concavity.

Example 2: Consider the function $g(x)=2x^4-8x+3$.

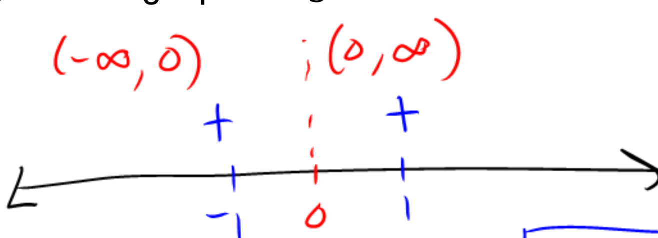
a. Discuss the concavity of the graph of g.

$g'(x) = 8x^3-8$

$g''(x) = 24x^2$

$0 = 24x^2$

$x = 0$



$g''(x) = 24x^2$

$g''(-1) = 24 > 0$

$g''(1) = 24 > 0$

$\boxed{\text{g is concave upward } (-\infty,0)\cup(0,\infty)\text{. g is never concave downward.}}$

b. Find all points of inflection.

$\boxed{\text{NONE}}$

Note: Since there's no change in concavity, there's no points of inflection.

THEOREM: SECOND DERIVATIVE TEST

Let f be a function such that $f'(c)=0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(x)>0$, then f has a **relative minimum** at $(c, f(c))$.
2. If $f''(x)<0$, then f has a **relative maximum** at $(c, f(c))$.
3. If $f''(x)=0$, the test FAILS and you need to run the FIRST DERIVATIVE TEST.

Example 3: Find all relative extrema. Use the Second Derivative Test where applicable.

a. $f(x)=x^3-5x^2+7x$

① Find C.N. [for f]

$f'(x) = 3x^2 - 10x + 7$

21

$-7 \quad -3$

-10

$f'(x) = 3x^2 - 7x - 3x + 7$

$f'(x) = x(3x-7) - 1(3x-7)$

$f'(x) = (3x-7)(x-1)$

$0 = (3x-7)(x-1)$

$3x-7=0$ or $x-1=0$

$x = \frac{7}{3}$ $\quad x = 1$

$c_1 = \frac{7}{3}$ and $c_2 = 1$

② Find $f''(x)$

$f''(x) = 6x - 10$

$f\left(\frac{7}{3}\right) = \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right)$

$= \frac{7}{3}\left[\frac{49}{9} - \frac{35}{3} + 7\right]$

$= \frac{7}{3}\left[\frac{49-105+63}{9}\right]$

$= \frac{7}{3}\left[\frac{7}{9}\right]$

$= \frac{49}{27}$

③ Plug c_1 and c_2 into $f''(x)$ to run the 2nd Derivative Test

$f''\left(\frac{7}{3}\right) = \overset{2}{\cancel{6}}\left(\frac{7}{\cancel{3}}\right) - 10 = 4 > 0$

curving upward at $c_1 = \frac{7}{3}$, so there's a relative minimum at $\left(\frac{7}{3}, f\left(\frac{7}{3}\right)\right) = \left(\frac{7}{3}, \frac{49}{27}\right)$.

$f''(1) = 6(1) - 10 = -4 < 0$

curving downward at $c_2 = 1$, so there's a relative max at $(1, f(1)) = (1, 3)$.

b. $f(x)=\dfrac{x}{x-1}$

$$f'(x)=\frac{1(x-1)-x(1)}{(x-1)^2}$$

$$\boxed{f'(x)=\frac{-1}{(x-1)^2}\neq 0 \text{ and at } x=1 \text{ there's a vertical asymptote, so there's no relative extrema.}}$$

Example 4: Sketch the graph of a function f having the given characteristics.

$f(0)=f(2)=0 \longrightarrow$ x-intercepts

$f'(x)>0$ if $x<1 \rightarrow f$ is $\uparrow$ on $(-\infty,1)$

$f'(1)=0 \rightarrow$ at $x=1$, f has a rel. max

$f'(x)<0$ if $x>1 \rightarrow f$ is $\downarrow$ on $(1,\infty)$

$f''(x)<0$ f is concave $\downarrow$ on $(-\infty,\infty)$