When you are done with your homework you should be able to…

- π Determine intervals on which a function is increasing or decreasing
- π Apply the First Derivative Test to find relative extrema of a function

Warm-up: Find the equation of the line tangent to the function $f(x)=\tan x$ at $x=\dfrac{3\pi}{4}$.

slope $f'(x) = \sec^2 x$

$f'(3\pi/4) = \left(\sec 3\pi/4\right)^2 = (-\sqrt{2})^2 = 2$

$\left(\sec\frac{3\pi}{4}\right)^2 = \left(\frac{1}{\cos 3\pi/4}\right)^2 = \left(\frac{1}{-\sqrt{2}/2}\right)^2 = \frac{4}{2} = 2$

y-coord $f\left(\frac{3\pi}{4}\right) = \tan\frac{3\pi}{4} = -1$

equation of tangent line $y-(-1) = 2(x-3\pi/4) \rightarrow \boxed{y+1 = 2(x-3\pi/4)}$

INCREASING AND DECREASING FUNCTIONS

A function is ___increasing___ if, as *x* moves to the right, its graph moves up, and is decreasing if its graph moves ___down___. A positive derivative implies that the function is ___increasing___ and a ___negative___ derivative implies that the function is decreasing.

DEFINITION OF INCREASING AND DECREASING ~~FUNCTIONS~~ Intervals

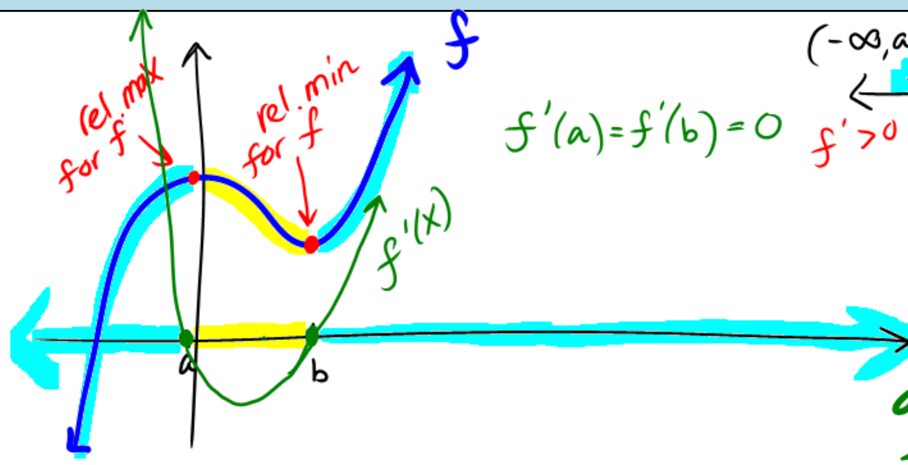
A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 ~~in the~~ in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

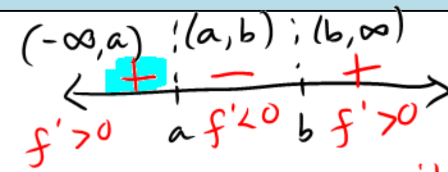
THEOREM: TEST FOR INCREASING AND DECREASING FUNCTIONS

Let f be a function that is continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b).

1. If $f'(x)>0$ for all x in (a,b), then f is **increasing** on (a,b).
2. If $f'(x)<0$ for all x in (a,b), then f is **decreasing** on (a,b).
3. If $f'(x)=0$ for all x in (a,b), then f is **constant** on (a,b).



$f'(a)=f'(b)=0$



$f'(x)$ is positive on $(-\infty,a)$.
$f'(x)$ is negative on (a,b)
$f'(x)$ is positive on (b,∞)

GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

Let f be continuous on the (a,b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use the test for increasing and decreasing ~~functions~~ intervals to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a,b) is replaced by an interval of the form $(-\infty,b)$, (a,∞), or $(-\infty,\infty)$.

Example 1: Identify the open intervals on which the function is increasing or decreasing.

$$f(x) = 27x - x^3$$

a. Find the critical numbers of f.

$f'(x) = 27 - 3x^2$

$0 = 27 - 3x^2$

$3x^2 = 27$

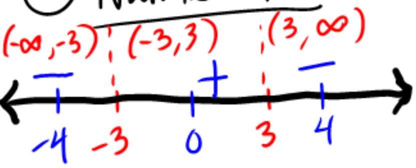
$\sqrt{x^2} = \sqrt{9}$

$\rightarrow x = \pm 3$

$c = \pm 3$

b. Run the test for increasing and decreasing intervals.

① Number line



$f'(x) = 27 - 3x^2$

② Test a value in each interval

$f'(-4) = 27 - 3(-4)^2 < 0$

$f'(0) = 27 - 3(0)^2 > 0$

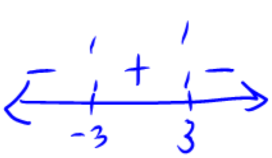
$f'(4) = 27 - 3(4)^2 < 0$

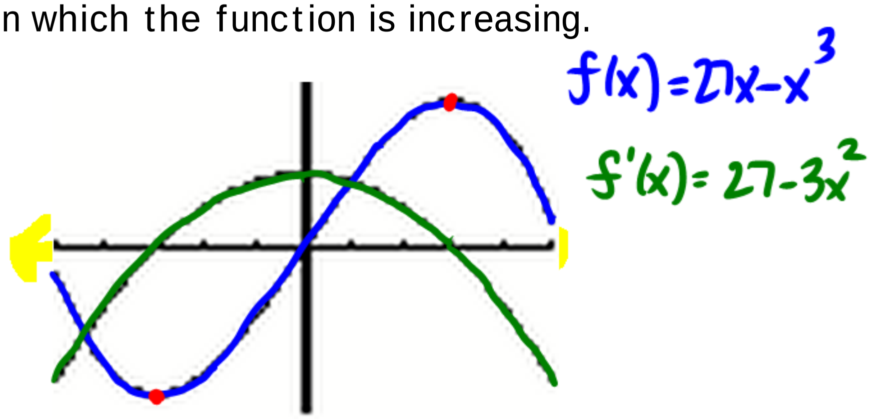
i. Find the open interval(s) on which the function is decreasing.

$\boxed{(-\infty,-3) \cup (3,\infty)}$

ii. Find the open interval(s) on which the function is increasing.

$\boxed{(-3,3)}$





Example 2: Identify the open intervals on which the function is increasing or decreasing.

$$f(x) = \cos^2 x - \cos x,\ \ 0 < x < 2\pi$$

a. Find the critical numbers of f.

$$f'(x) = -2\cos x \sin x + \sin x$$

$$0 = -\sin x(2\cos x - 1)$$

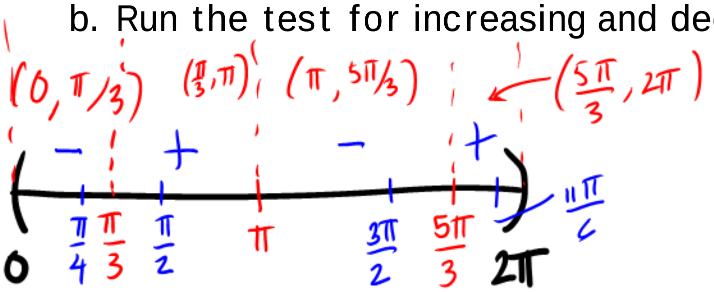
$$-\sin x = 0 \qquad 2\cos x - 1 = 0$$

$$\sin x = 0 \qquad \cos x = \frac{1}{2}$$

$$x = \pi \qquad x = \pi/3,\ 5\pi/3$$

$$\boxed{c = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}}$$

b. Run the test for increasing and decreasing intervals.



$$f'(x) = -\sin x(2\cos x - 1)$$

$$f'(\pi/4) = -\frac{\sqrt{2}}{2}\left(\cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right) - 1\right)$$

$$= -\frac{\sqrt{2}}{2}(\sqrt{2} - 1) < 0$$

$$f'(\pi/2) = -1\,(2 \cdot 0 - 1) > 0$$

$$f'(3\pi/2) = -(-1)(2 \cdot 0 - 1) < 0$$

$$f'(11\pi/6) = -(-\tfrac{1}{2})\left(\cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} - 1\right) > 0$$

i. Find the open interval(s) on which the function is decreasing.

$$\boxed{\left(0, \frac{\pi}{3}\right) \cup (\pi, 5\pi/3)}$$

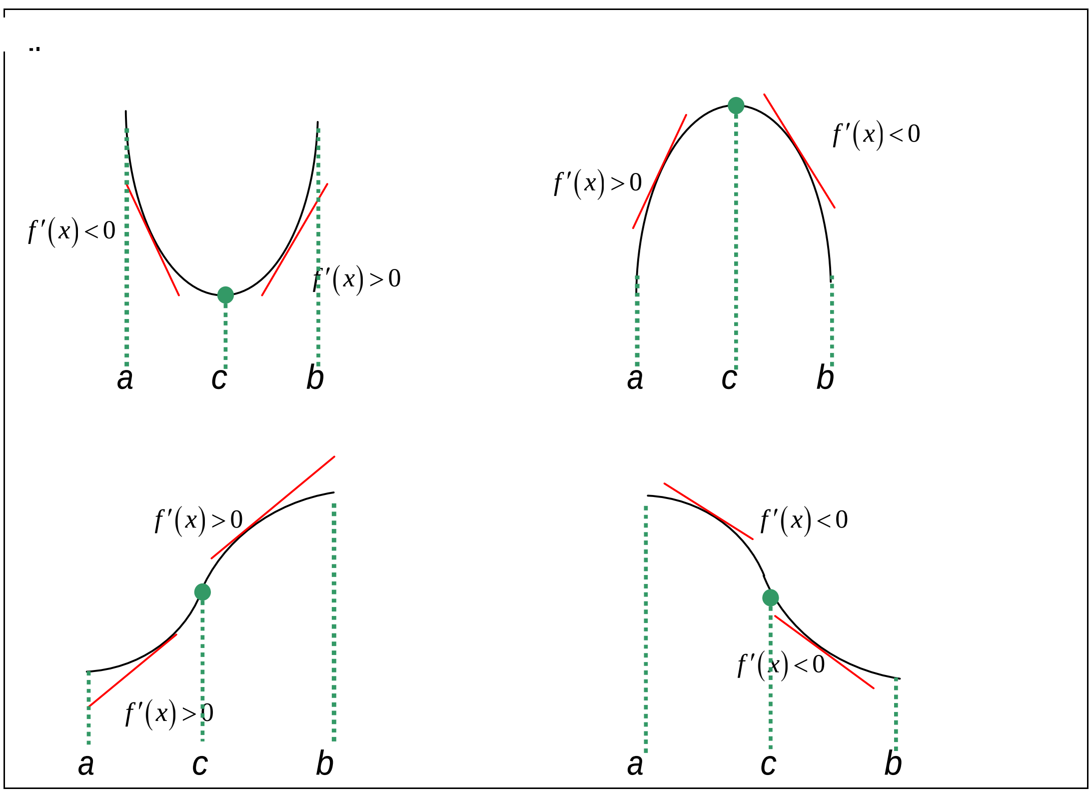
ii. Find the open interval(s) on which the function is increasing.

$$\boxed{\left(\frac{\pi}{3}, \pi\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)}$$

THEOREM: THE FIRST DERIVATIVE TEST

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum** at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a **relative maximum** at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum or relative maximum.



Example 3: Consider the function $g(x) = x^{2/3} - 4$.

$g(x) = (x^{1/3})^2 - 4$

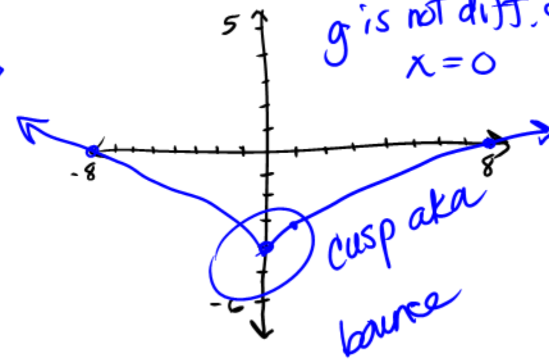
a. Find the critical numbers of g.

$g'(x) = \frac{2}{3} x^{-1/3}$

$g'(x) = \frac{2}{3x^{1/3}}$

$0 = \frac{2}{3x^{1/3}}$ $\quad 0 = 2$ false $\quad$ no zeros

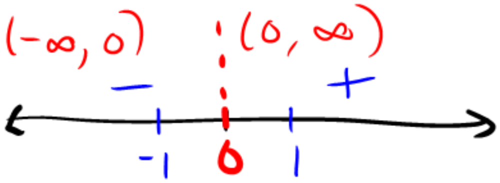
$\boxed{c = 0}$



g is not diff. at $x=0$

cusp aka bounce

b. Run the test for increasing and decreasing intervals.



$g'(x) = \frac{2}{3x^{1/3}}$

$g'(-1) = \frac{2}{3\sqrt[3]{-1}} < 0$

$g'(1) = \frac{2}{3\sqrt[3]{1}} > 0$

i. Find the open interval(s) on which the function is decreasing.

$\boxed{(-\infty, 0)}$

ii. Find the open interval(s) on which the function is increasing.

$\boxed{(0, \infty)}$

c. Apply the First Derivative Test.

i. Identify all relative minima.

$\boxed{-4 \text{ / occurs at } (0,-4)}$

$(c, g(c)) = (0, g(0))$
$= (0, -4)$

ii. Identify all relative maxima.

NONE

Example 4: The graph of a function f is given. Sketch a graph of the derivative of f.