When you are done with your homework you should be able to…

- π Understand the definition of a function on an interval
- π Understand the definition of relative extrema of a function on an open interval
- π Find extrema on a closed interval

Warm-up: Determine the point(s) at which the graph of $f(x)=2x^2-8x+5$ has a horizontal tangent.

$f'(x)=4x-8$

$0=4x-8$

$8=4x$

$2=x$

$f(2)=-3$

At $(2,-3)$ f has a horizontal tangent.

EXTREMA OF A FUNCTION

In calculus, much effort is devoted to analyzing the behavior of a function f on an interval I. Does f have a minimum value or maximum value on I? Where is f decreasing? Where is f increasing? In this chapter, you will learn how differentiation can be used to answer these questions.

DEFINITION OF EXTREMA

Let f be defined on an open interval I containing c.

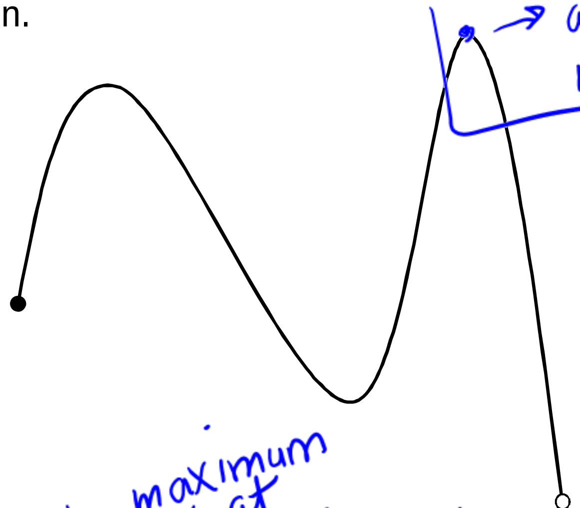
1. $f(c)$ is the **minimum of** f **on** I if $f(c) \le f(x)$ for all x in I.

2. $f(c)$ is the **maximum of** f **on** I if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval.

Let's check out the functions below. Identify the maximum and minimum of each function.

a.



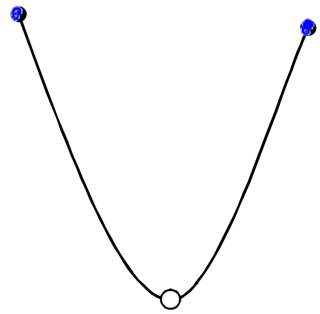
→ absolute maximum or maximum or global maximum

no minimum.

b.

the maximum occurs at these 2 points

(x_1, y_0) (x_2, y_0)



no absolute minimum

THEOREM: THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the closed interval.

DEFINITION OF RELATIVE EXTREMA

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum of** f, or you can say that f has a **relative maximum at** $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum of** f, or you can say that f has a **relative minimum at** $(c, f(c))$.

Example 1: Find the value of the derivative at the extremum $(0,1)$, for the function $f(x) = \cos\frac{\pi x}{2}$.

$$f'(x) = \left(-\sin\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}$$

$$f'(0) = \left(-\sin\frac{\pi \cdot 0}{2}\right)\frac{\pi}{2} = 0$$

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c)=0$ **OR** if f is not differentiable at c, then c is a **critical number**.

Example 2: Find any critical numbers of the following functions.

a. $f(x)=2x^2-8x+5$

$f'(x)=4x-8$

$0=4x-8$

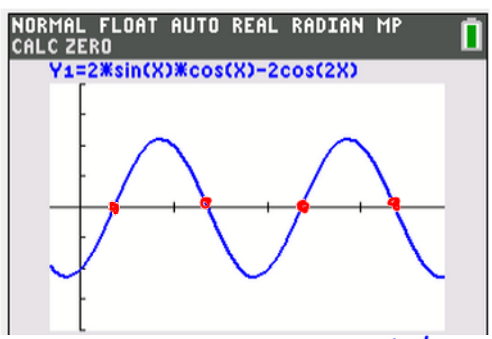
$x=2$

$\boxed{c=2}$

b. $g(x)=\sin^2 x-\sin 2x,\ [0,2\pi)$

$g'(x)=2\sin x\cos x-2\cos 2x$

$0=2(\sin x\cos x-\cos 2x)$



$x\approx 0.5536,\ 2.1244,\ 3.6952,\ 5.2660$

$\boxed{c\approx 0.5536,\ 2.1244,\ 3.6952,\ 5.2660}$

c. $s(t)=\dfrac{3t}{t^2-4}$

$s'(t)=\dfrac{3(t^2-4)-3t(2t)}{(t^2-4)^2}$

$s'(t)=\dfrac{3t^2-12-6t^2}{(t^2-4)^2}$

$s'(t)=\dfrac{-3(t^2+4)}{(t^2-4)^2}$

$0=-3(t^2+4)$

$0=t^2+4$

$t^2=-4 \rightarrow t=\pm 2i$

$t\notin\mathbb{R}$

$\boxed{\text{no critical \#'s}}$

THEOREM: RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x=c$, then c is a critical number of f.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a,b]$, use the following steps.

1. Find the critical numbers of f in (a,b).
2. Evaluate f at each critical number in (a,b).
3. Evaluate f at each endpoint of $[a,b]$.
4. The least of these numbers is the minimum. The greatest is the maximum.

Example 3: Locate the absolute extrema of the following functions function on the closed interval.

1) CN

a. $f(x)=3\cos x,\ [0,2\pi]$

$f'(x) = -3\sin x$

$0 = -3\sin x$

$0 = \sin x$

$x = 0, \pi, 2\pi \rightarrow [0, 2\pi]$

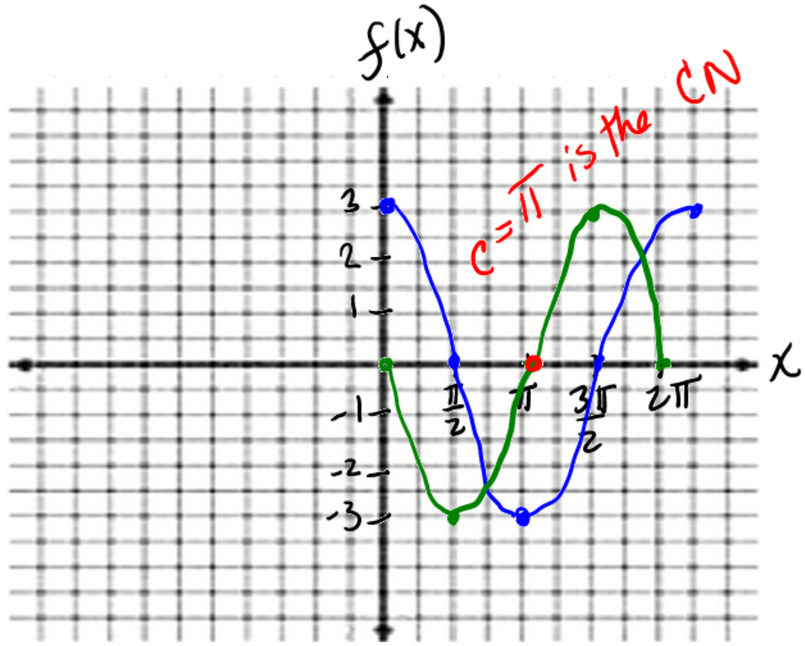
$c = \pi \rightarrow (0, 2\pi)$

2) $f(c)$

$f(\pi) = 3\cos\pi = -3$

3) evaluate the endpoints at f

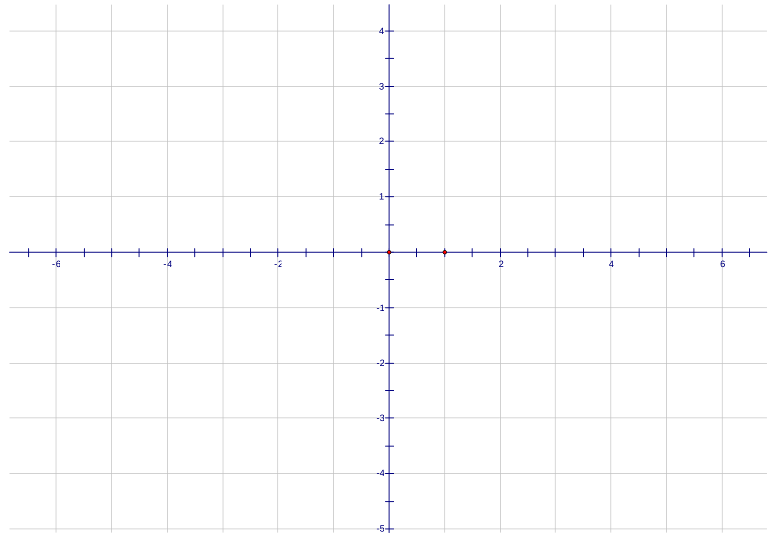
$f(0) = 3,\ f(2\pi) = 3$



4) Conclusion

The absolute minimum is -3 and occurs at $(\pi, -3)$.
The absolute maximum is 3 and occurs at $(0, 3)$ and $(2\pi, 3)$.

b. $g(t)=\dfrac{t}{t-2}, \; [3,5]$



Example 4: A retailer has determined that the cost C of ordering and storing x units of a product is $C = 2x + \dfrac{300,000}{x}, \; 1 \le x \le 300.$ The delivery truck can bring at most 300 units per order.

a. Find the order size that will minimize cost.

$C'(x) = 2 - \dfrac{300000}{x^2}$

$0 = 2 - \dfrac{300000}{x^2}$

$\dfrac{300000}{x^2} = 2$

$2x^2 = 300000$

$\sqrt{x^2} = \sqrt{150000}$

$x \doteq \pm 387$

no C.N. $\rightarrow [1, 300]$

$C(1) = 300,002$

$C(300) = 1,600$

An order size of 300 units yields the minimum cost of \$1,600.

b. Could the cost be decreased if the truck were replaced with one that could bring at most 400 units? Explain. $[1, 400]$

$c = 387$

$C(387) \doteq 1,549$

$C(400) \doteq 1550$

Yes. 387 units would yield a cost of \$1,549.