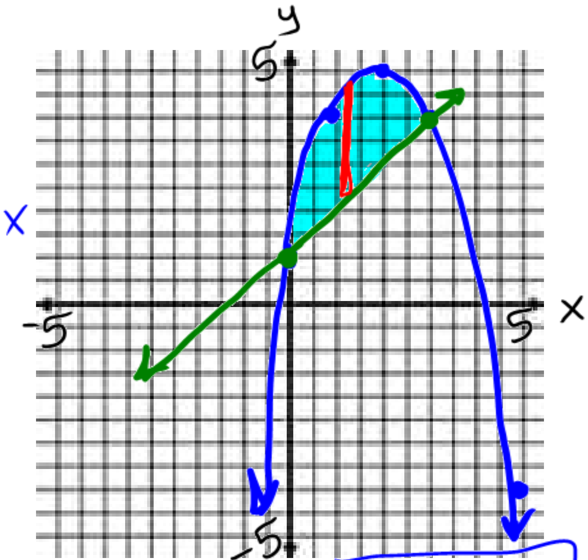
When you are done with your homework you should be able to…

- π Find the volume of a solid of revolution using the disk method
- π Find the volume of a solid of revolution using the washer method
- π Find the volume of a solid with known cross sections

Warm-up: Sketch the region bounded by the graphs of the algebraic functions and find the area of the region.

a. $f(x) = -x^2 + 4x + 1, \quad g(x) = x + 1$



Bounds

$-x^2 + 4x + 1 = x + 1$

$0 = x^2 - 3x$

$0 = x(x-3)$

$x = 0$ or $x = 3$

Area:

$\int_0^3 \left[(-x^2+4x+1) - (x+1)\right] dx$

$= \int_0^3 (-x^2 + 3x)\, dx$

$= -\frac{x^3}{3} + \frac{3}{2}x^2 \Big|_0^3$

$= -\frac{27}{3} + \frac{3}{2} \cdot 9$

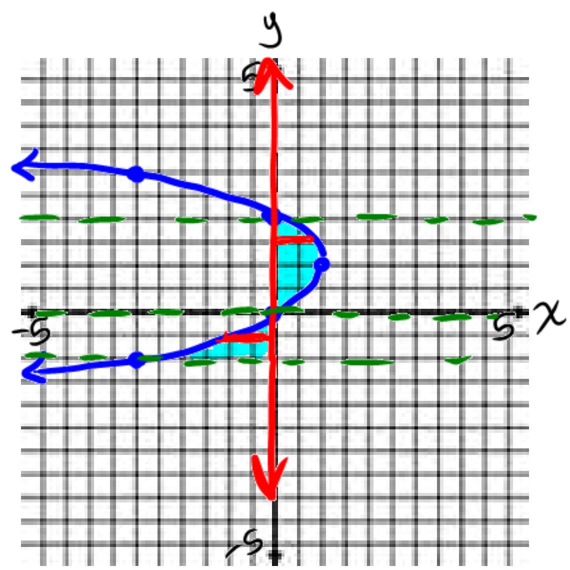
$= -9 + \frac{27}{2}$

$= \frac{-18+27}{2}$

$= \boxed{\frac{9}{2} \text{ sq. units}}$

$x = 0$

b. $f(y) = y(2-y), \quad g(y) = 0, \quad y = -1, \quad y = 2$



Bounds

$y(2-y) = 0$

$y = 0$ or $y = 2$

Area

$A = \int_{-1}^{0} \left[0 - (2y - y^2)\right] dy + \int_0^2 \left[(2y - y^2) - 0\right] dy$

$A = \int_{-1}^{0} (-2y + y^2)\, dy + \int_0^2 (2y - y^2)\, dy$

$A = \left(-y^2 + \frac{y^3}{3}\right)\Big|_{-1}^{0} + \left(y^2 - \frac{y^3}{3}\right)\Big|_0^2$

$= \left[0 - \left(-1 - \frac{1}{3}\right)\right] + \left[\left(4 - \frac{8}{3}\right) - 0\right]$

$= \frac{4}{3} + \frac{4}{3}$

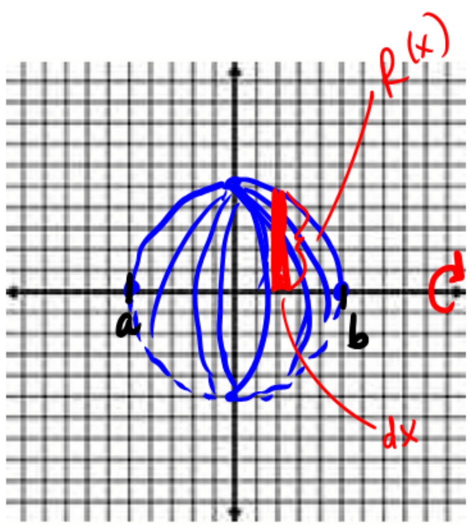
$= \boxed{8/3 \text{ sq. units}}$

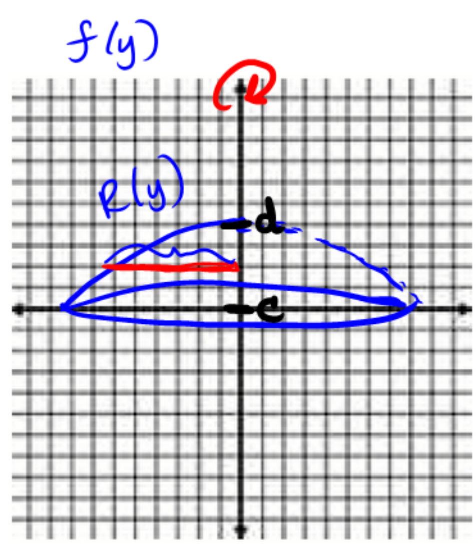
THE DISK METHOD

An important application of the __definite__ integral is its use in finding the __volume__ of a three-dimensional solid—one whose __cross__ sections are __rotated__.

Solids of revolution are used commonly in engineering and manufacturing. Some examples are __________, __________, __________, __________, and __________.

If a __region__ in the __plane__ is __rotated__ about a __line__, the resulting __shape__ is a **solid of revolution**, and the line is called the **axis of revolution**.





THE DISK METHOD

To find the volume of a solid of revolution with the **disk method** use one of the following:

Horizontal Axis of Revolution

$$V=\pi\int_a^b\left[R(x)\right]^2dx$$

Vertical Axis of Revolution

$$V=\pi\int_c^d\left[R(y)\right]^2dy$$

Example 1: Find the volume of the solids generated by revolving the regions bounded by the graphs of the equations about the given line.

a) $y=2x^2,\ y=0,\ x=2,$ about the $x-$axis.

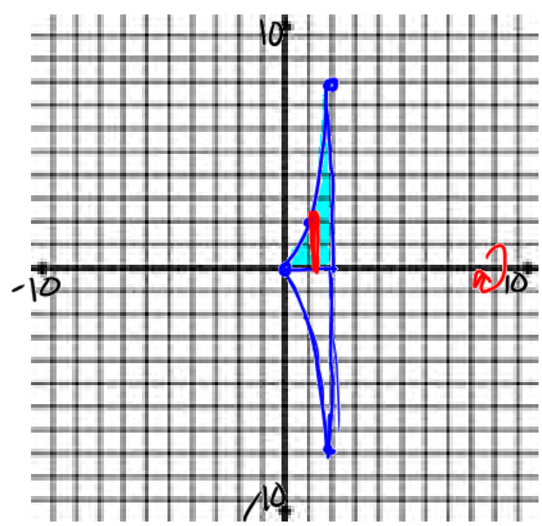
$R(x)=(2x^2)-0=2x^2$

$V=\pi\int_0^2(2x^2)^2\,dx$

$V=4\pi\int_0^2 x^4\,dx$

$V=\frac{4\pi}{5}x^5\Big|_0^2$

$V=\frac{128\pi}{5}$ units cubed



b) $y = 2x^2,\ y = 0,\ x = 2,$ about the $y-$axis.

Need to find R(y)

$y = 2x^2$

$\frac{y}{2} = x^2$

$f(y) = x = \sqrt{\frac{y}{2}}$

$(2\sqrt{2})^3 = 8 \cdot 2\sqrt{2}$

$R(y) = 2 - \sqrt{\frac{y}{2}}$

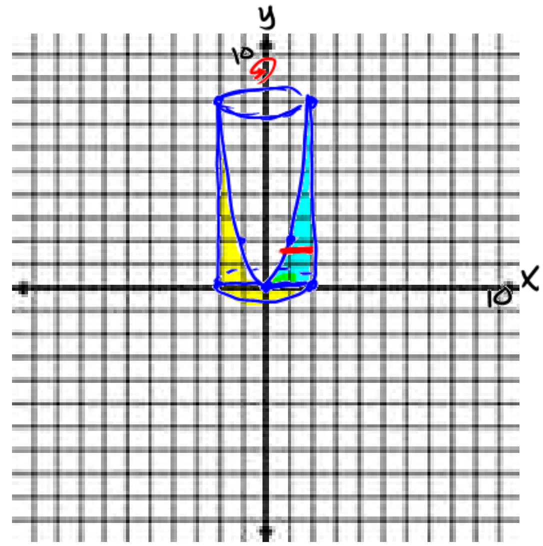
$V = \pi \int_0^8 \left[2 - \left(\frac{y}{2}\right)^{1/2}\right]^2 dy$

$V = \pi \left[\int_0^8 \left(4 - 2(2)\left(\frac{y}{2}\right)^{1/2} + \frac{y}{2}\right) dy \right.$

$V = \pi \int_0^8 \left(4 - \frac{4}{\sqrt{2}} y^{1/2} + \frac{y}{2}\right) dy$

$V = \pi \left(4y - \frac{8}{3\sqrt{2}} y^{3/2} + \frac{y^2}{4}\right|_0^8 \rightarrow V = \pi\left(32 - \frac{8}{3\sqrt{2}} 16\sqrt{2} + 16\right)$

$V = \pi\left(\frac{144 - 128}{3}\right) \rightarrow V = \frac{16\pi}{3}$ units cubed



THE WASHER METHOD

The disk method can be extended to cover solids of revolution with ______holes______

by replacing the representative ________________ with a representative

__________________________.

THE WASHER METHOD

To find the volume of a solid of revolution with the **washer method** use one of the following:

Horizontal Axis of Revolution

$$V = \pi \int_a^b \left(\left[R(x)\right]^2 - \left[r(x)\right]^2 \right) dx$$

Vertical Axis of Revolution

$$V = \pi \int_c^d \left(\left[R(y)\right]^2 - \left[r(y)\right]^2 \right) dy$$

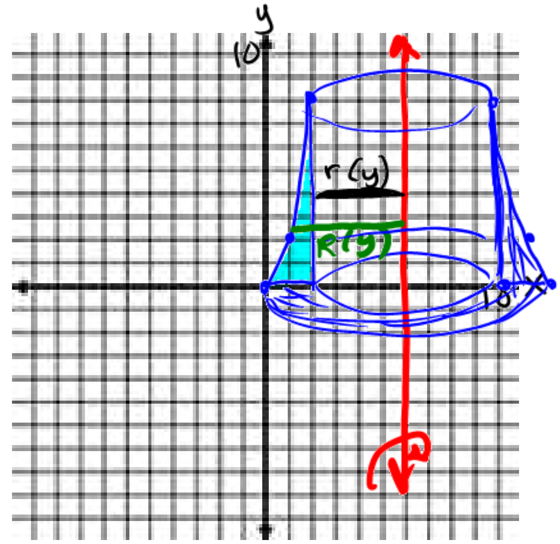
Example 2: Find the volume of the solids generated by revolving the regions bounded by the graphs of the equations about the given line.

a) $y = 2x^2,\ y = 0,\ x = 2,$ about the line $x = 6$.

$r(y) = (6-2) = 4$

$R(y) = 6 - \left(\frac{y}{2}\right)^{1/2}$

$V = \pi \int_0^8 \left[\left(6 - \left(\frac{y}{2}\right)\right)^2 - (4)^2 \right] dy$



b) $y = \cos x,\ y = 1,\ x = 0,\ x = \frac{\pi}{2}$ about the line $y = 2$.

$r(x) = (2-1) = 1$

$R(x) = 2 - \cos x$

$V = \pi \int^{\pi/2} \left[(2-\cos x)^2 - (1)^2 \right] dx$

$V = \pi \int_0^{\pi/2} \left(4 - 2\cos x + \cos^2 x - 1\right) dx$

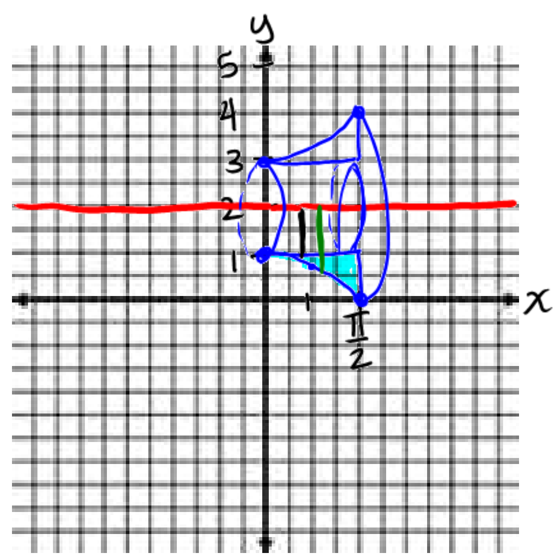
$V = \pi \int_0^{\pi/2} (3 - 2\cos x)\, dx + \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x)\, dx$

$V = \pi \left(3x - 2\sin x\right|_0^{\pi/2} + \frac{\pi}{2}\left(x + \frac{1}{2}\sin 2x\right|_0^{\pi/2}$

$V = \frac{3\pi^2}{2} - 2\pi + \frac{\pi^2}{4} + 0$

$V = \frac{7\pi^2}{4} - \frac{8\pi}{4}$

$\boxed{V = \frac{\pi}{4}(7\pi - 8) \text{ cubic units}}$



SOLIDS WITH KNOWN CROSS SECTIONS

With the disk method, you can find the ________________________ of a solid

having a ______________________ cross section whose area is _____________.
This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section. Some common cross sections

are ____________________, _________________________,

______________________, _________________________, and

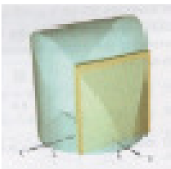
________________________________.

VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the *x*-axis,
$$V = \int_a^b A(x)dx$$
2. For cross sections of area $A(y)$ taken perpendicular to the *y*-axis,
$$V = \int_c^d A(y)dy$$

Example 3: Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$ with the indicated cross sections taken perpendicular to the *x*-axis.

a) Squares



b) Semicircles