When you are done with your homework you should be able to…

- π Integrate functions whose antiderivatives involve inverse trigonometric functions
- π Use the method of completing the square to integrate a function
- π Review the basic integration rules involving elementary functions

Warm-up:

1. Differentiate the following functions with respect to x.

a. $y = \arctan\frac{x}{2} - \frac{1}{2\left(x^2+4\right)}$

b. $\frac{\partial}{dx}\left(\arctan(xy)\right) = \frac{\partial}{dx}\left(\arcsin(x+y)\right)$.

$$\frac{\frac{\partial}{dx}(xy)}{1+(xy)^2} = \frac{\frac{\partial}{dx}(x+y)}{\sqrt{1-(x+y)^2}}$$

$$\frac{1y+xy'}{1+x^2y^2} = \frac{1+y'}{\sqrt{1-(x+y)^2}}$$

$$(y+xy')\sqrt{1-(x+y)^2} = (1+y')(1+x^2y^2)$$

$$y\sqrt{1-(x+y)^2} + y'x\sqrt{1-(x+y)^2} = 1+x^2y^2+y'+y'x^2y^2$$

$$y\sqrt{1-(x+y)^2} - 1 - x^2y^2 = y'\left(1+x^2y^2 - x\sqrt{1-(x+y)^2}\right)$$

$$\boxed{y' = \frac{y\sqrt{1-(x+y)^2} - 1 - x^2y^2}{1+x^2y^2 - x\sqrt{1-(x+y)^2}}}$$

2. Complete the square.

a. $3+4x-x^2$ $= -x^2+4x+3$

$= -\left(x^2-4x+(-2)^2\right)+3+4$

$= \boxed{7-(x-2)^2}$

b. $2x^2-6x+9$ $= 2\left(x^2-3x+\left(-\frac{3}{2}\right)^2\right)+9-\frac{9}{2}$

$= \boxed{2\left(x-\frac{3}{2}\right)^2+\frac{9}{2}}$

What did you notice about the derivatives of the inverse trigonometric functions?

$$\frac{d}{dx}\arccos u = -\frac{d}{dx}\arcsin u$$

$$\frac{d}{dx}\operatorname{arccot} u = -\frac{d}{dx}\arctan u$$

$$\frac{d}{dx}\operatorname{arccsc} u = -\frac{d}{dx}\operatorname{arcsec} u$$

THEOREM: INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a>0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C$

2. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$

3. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+C$

$a^2 = 1^2$

$a = 1$

$u = x$

$du = dx$

Example 1: Find or evaluate the integral.

a. $\int \frac{dx}{x\sqrt{x^2-1}} = \frac{1}{1}\text{arcsec}\frac{|x|}{1} + C$

$= \boxed{\text{arcsec}|x| + C}$

$\int \frac{u\,du}{\sqrt{u^2-1}}$

b. $\int \frac{x\,dx}{\sqrt{x^2-1}} = \int x(x^2-1)^{-1/2}\,dx$

$u = x^2-1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int \cancel{x}\, u^{-1/2}\left(\frac{du}{2\cancel{x}}\right)$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\frac{u^{1/2}}{1/2} + C$

$= \boxed{(x^2-1)^{1/2} + C}$

c. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin\frac{x}{1} + C$

$= \boxed{\arcsin x + C}$

d. $\int \frac{dx}{x \ln x} = \int \frac{\cancel{x} du}{\cancel{x} u}$

$= \int \frac{du}{u}$

$= \ln|u| + C$

$= \boxed{\ln|\ln x| + C}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x du$

e. $\int \frac{(\ln x)^2 dx}{x} = \int \frac{u^2 (\cancel{x} du)}{\cancel{x}}$

$= \int u^2 du$

$= \frac{u^3}{3} + C$

$= \boxed{\frac{(\ln x)^3}{3} + C}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x du$

$(\ln x)^3 = (\ln x)(\ln x)(\ln x)$

$\ln x^3 = \ln(x \cdot x \cdot x)$

$= \ln x + \ln x + \ln x$

$= 3 \ln x$

f. $\int \ln x dx$

Need Integration By Parts

Calc II, 8.2

Example 2: Find the integral by completing the square.

a. $\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(3)^2+(x+2)^2}$

$= \int \frac{du}{(3)^2+(u)^2}$

$= \frac{1}{3}\arctan\frac{u}{3} + C$

$= \boxed{\frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C}$

$x^2+4x+13 = (x^2+4x+(2)^2)+13-4$

$= 9+(x+2)^2$

$=(3)^2+(x+2)^2$

$a=3$

$u=x+2$

$\frac{du}{dx}=1$

$dx=du$

$\cancel{\int (x^2+4x+13)^{-1}dx}$

$\cancel{u=x^2+4x+13}$

$\cancel{\frac{du}{dx}=2x+4}$

$\cancel{dx=\frac{du}{2x+4}}$

b. $\int \frac{dx}{x\sqrt{x^4-4}} = \int \frac{dx}{x\sqrt{(x^2)^2-(2)^2}}$

$= \int \frac{1}{x\sqrt{u^2-(2)^2}} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int \frac{du}{x^2\sqrt{u^2-2^2}}$

$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-2^2}}$

$= \frac{1}{2}\cdot\frac{1}{2}\operatorname{arcsec}\frac{|u|}{2} + C$

$= \boxed{\frac{1}{4}\operatorname{arcsec}\left(\frac{x^2}{2}\right) + C}$

$u=x^2 \rightarrow x^2=u$

$\frac{du}{dx}=2x$

$dx=\frac{du}{2x}$

$a=2$

c. $\int \frac{2dx}{\sqrt{-x^2+4x}} = 2\int \frac{du}{\sqrt{(2)^2-(u)^2}}$

$-x^2+4x = -(x^2-4x+(-2)^2)+4$

$= 4-(x-2)^2$

$= (2)^2-(x-2)^2$

$a=2, \quad u=x-2$

$\frac{du}{dx}=1$

$dx=du$

$= 2\arcsin\frac{u}{2}+C$

$= \boxed{2\arcsin\left(\frac{x-2}{2}\right)+C}$

d. $\int \frac{2x-5}{x^2+2x+2}dx = \int \frac{2x+2}{x^2+2x+2}dx + \int \frac{-7}{x^2+2x+2}dx$

$u=x^2+2x+2$

$\frac{du}{dx}=2x+2$

$dx=\frac{du}{2x+2}$

$(x^2+2x+(1)^2)+2-1$

$=(x+1)^2+1$

$=(1)^2+(x+1)^2$

$a=1, \quad u=x+1$

$\frac{du}{dx}=1$

$dx=du$

$= \int \frac{\cancel{2x+2}}{u}\cdot\frac{du}{\cancel{2x+2}} - 7\int\frac{du}{(1)^2+(u)^2}$

$= \ln|u| - 7\left(\frac{1}{1}\arctan\frac{u}{1}\right)+C$

$= \boxed{\ln\left|x^2+2x+2\right| - 7\arctan(x+1)+C}$

e. $\int \frac{x}{\sqrt{9+8x^2-x^4}}dx = \int \frac{\cancel{x}}{\sqrt{9+8u-u^2}} \cdot \frac{du}{2\cancel{x}}$

$u=x^2,\ \frac{du}{dx}=2x \rightarrow dx=\frac{du}{2x}$

$-u^2+8u+9=-(u^2-8u+(-4)^2)+9+16$

$=25-(u-4)^2$

$=(5)^2-(u-4)^2$

$a=5$

$= \frac{1}{2}\int \frac{du}{\sqrt{(5)^2-(u-4)^2}}$

$= \frac{1}{2}\arcsin \frac{u-4}{5} + C$

$= \boxed{\frac{1}{2}\arcsin \frac{x^2-4}{5} + C}$

f. $\int_1^3 \frac{1}{\sqrt{x}(1+x)}dx = \int_1^{\sqrt{3}} \frac{1}{\cancel{\sqrt{x}}((1)^2+(u)^2)} \cdot 2\cancel{\sqrt{x}}\,du$

$u=\sqrt{x},\ u^2=x$

$\frac{du}{dx}=\frac{1}{2\sqrt{x}}$

$dx=2\sqrt{x}\,du$

$u(3)=\sqrt{3}$

$u(1)=\sqrt{1}=1$

$= 2\int_1^{\sqrt{3}} \frac{du}{(1)^2+(u)^2}$

$= 2\left(\frac{1}{1}\arctan \frac{u}{1}\right|_1^{\sqrt{3}}$

$= 2\left(\arctan\sqrt{3} - \arctan 1\right)$

$= 2\left(\pi/3 - \pi/2\right)$

$= \cancel{2}^{1}\left(\frac{2\pi-3\pi}{\cancel{6}_{3}}\right)$

$= \boxed{-\frac{\pi}{3}}$

g. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}dx = \int_0^1 \frac{\cancel{\cos x}}{1+u^2} \cdot \frac{du}{\cancel{\cos x}}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \frac{1}{1}\arctan \frac{u}{1} \Big|_0^1$

$= \arctan 1 - \arctan 0$

$a = 1$

$u(\pi/2) = \sin \pi/2 = 1$

$u(0) = \sin 0 = 0$

$= \frac{\pi}{4} - 0$

$= \boxed{\frac{\pi}{4}}$

Example 3: Find the area of the region bound by the graphs of

$y = \frac{4e^x}{1+e^{2x}}$, $x = 0$, $y = 0$ and $x = \ln\sqrt{3}$.