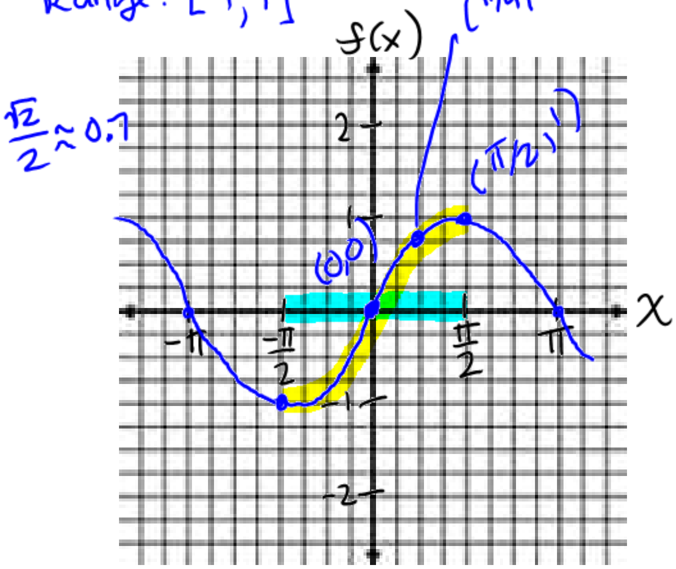
When you are done with your homework you should be able to…

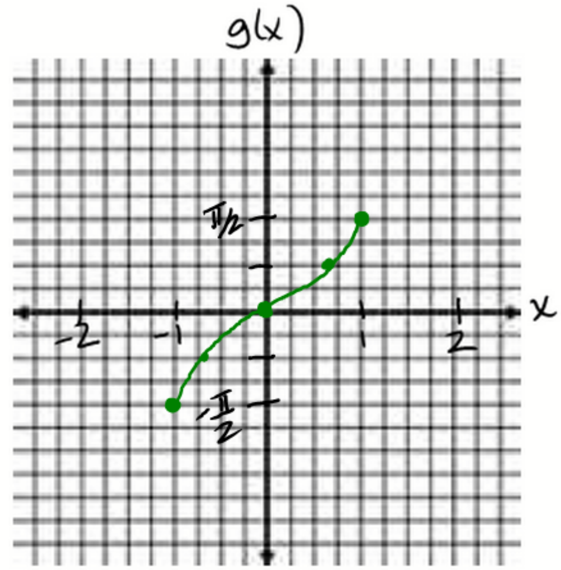
- π Develop properties of the six inverse trigonometric functions
- π Differentiate an inverse trigonometric function
- π Review the basic differentiation rules for elementary functions

Warm-up: Draw the following graphs by hand from $[-\pi,\pi]$. List the domain and range in interval notation.

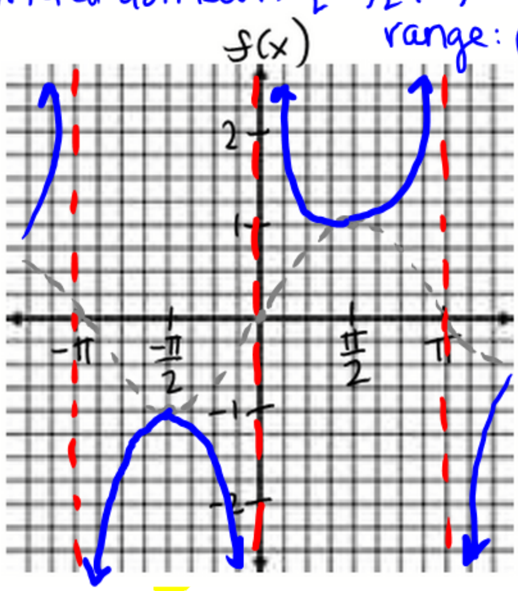
Rest. 1. Graph $f(x)=\sin x$.
Domain: $[-\pi/2, \pi/2]$
Range: $[-1,1]$



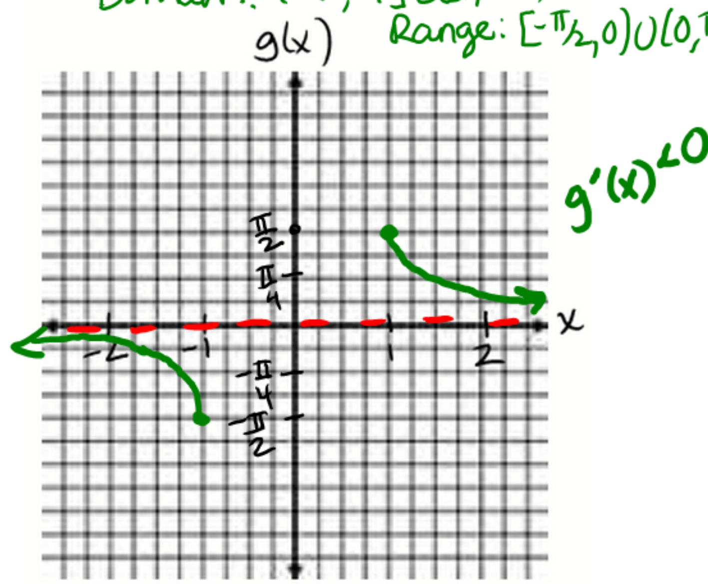
2. Graph $g(x)=\arcsin x$.
Domain: $[-1,1]$ Range: $[-\pi/2, \pi/2]$



3. Graph $f(x)=\csc x$.
restricted domain: $[-\pi/2, 0) \cup (0, \pi/2]$
range: $(-\infty,-1] \cup [1,\infty)$



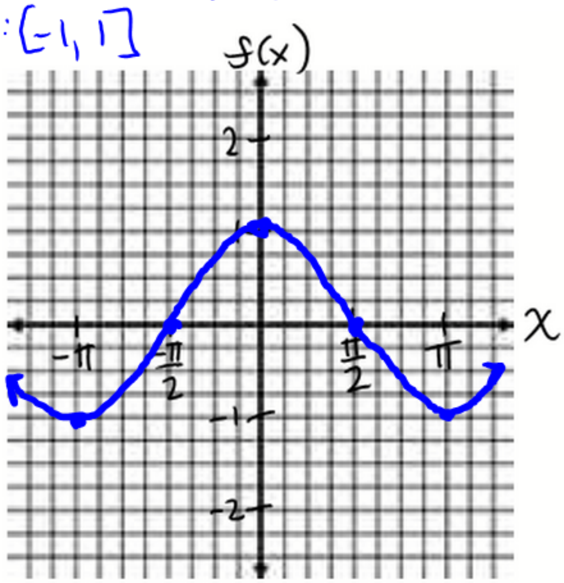
4. Graph $g(x)=\operatorname{arc}\csc x$.
Domain: $(-\infty,-1] \cup [1,\infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



5. Graph $f(x) = \cos x$.

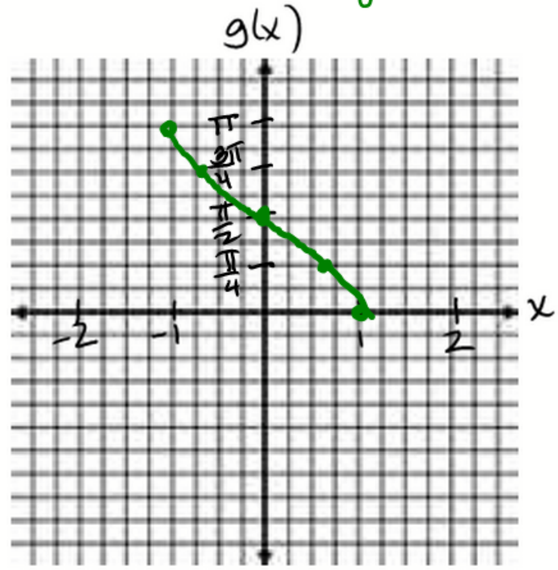
rest. dom: $[0, \pi]$

range: $[-1, 1]$



6. Graph $g(x) = \arccos x$.

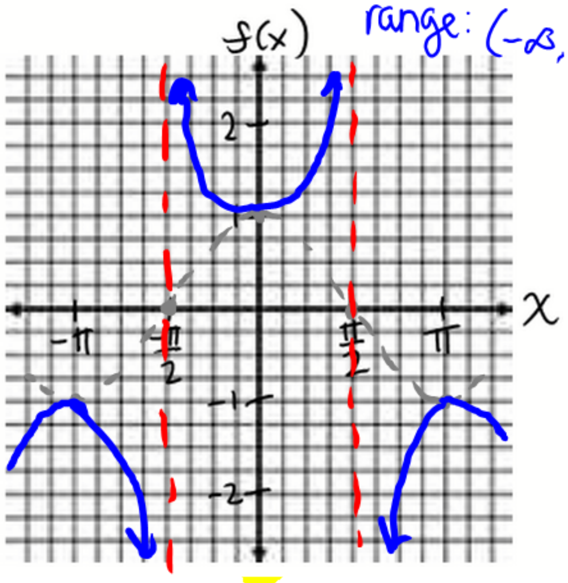
dom: $[-1, 1]$, range: $[0, \pi]$



7. Graph $f(x) = \sec x$.

rest. dom: $[0, \pi/2) \cup (\pi/2, \pi]$

range: $(-\infty, -1] \cup [1, \infty)$

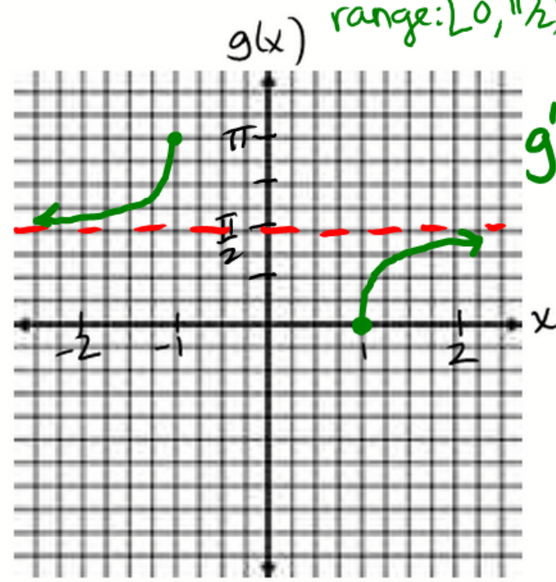


8. Graph $g(x) = \operatorname{arc}\sec x$.

dom: $(-\infty, -1] \cup [1, \infty)$

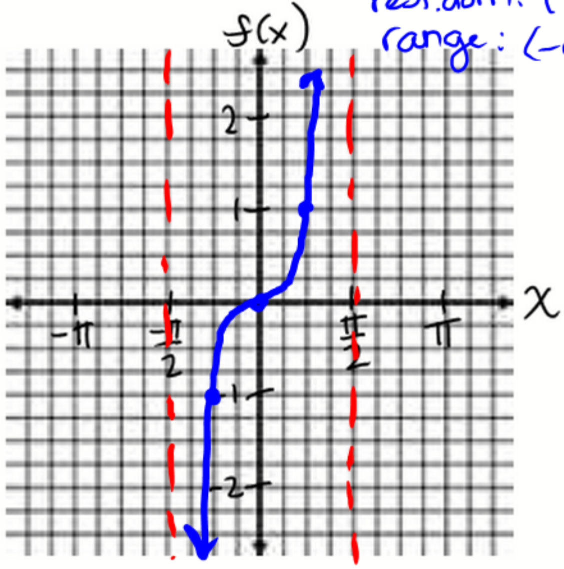
range: $[0, \pi/2) \cup (\pi/2, \pi]$

$g'(x) > 0$



9. Graph $f(x) = \tan x$.

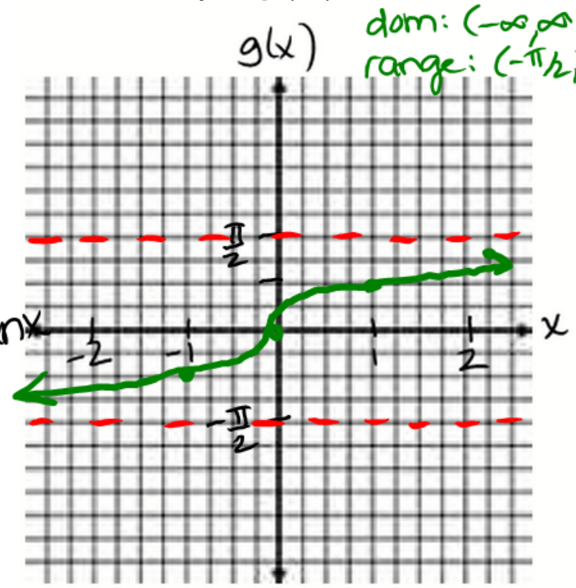
rest. dom: $(-\pi/2, \pi/2)$
range: $(-\infty, \infty)$



10. Graph $g(x) = \arctan x$.

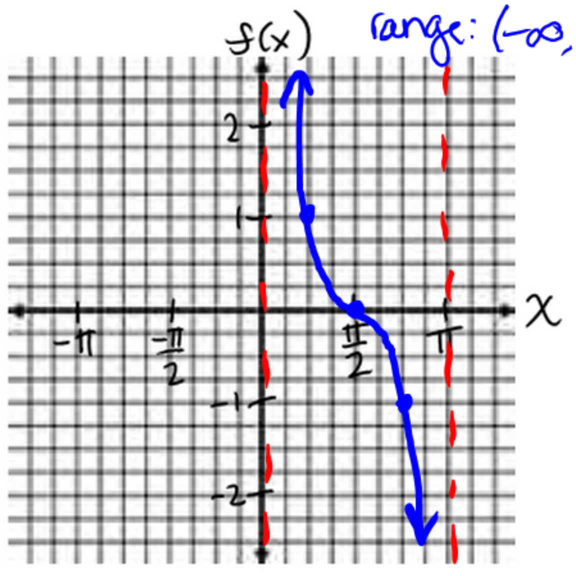
dom: $(-\infty, \infty)$
range: $(-\pi/2, \pi/2)$

$\lim_{x \to \infty} \arctan x = \boxed{\frac{\pi}{2}}$



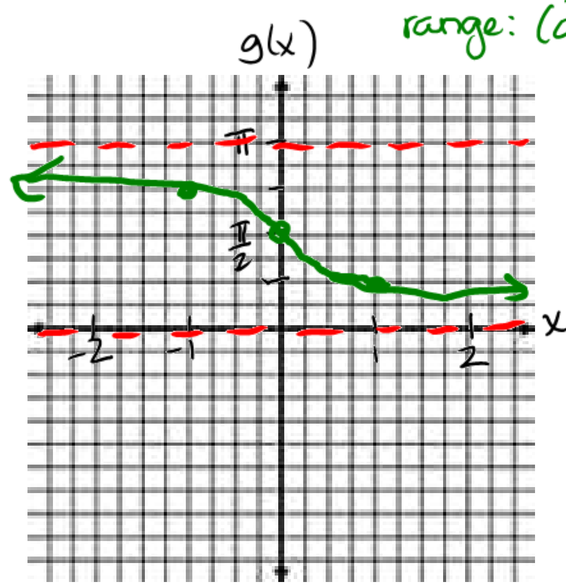
11. Graph $f(x) = \cot x$.

rest. dom: $(0, \pi)$
range: $(-\infty, \infty)$



12. Graph $g(x) = \text{arc}\cot x$.

dom: $(-\infty, \infty)$
range: $(0, \pi)$



PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

If $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\frac{\pi}{2} < y < \frac{\pi}{2}$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \le \pi$, then

$$\sec(\operatorname{arc}\sec x) = x \quad \text{and} \quad \operatorname{arc}\sec(\sec y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \quad \text{and} \quad \arccos(\cos y) = y.$$

If $\cancel{-\frac{\pi}{2} < y < \frac{\pi}{2}}$, then

$0 < y < \pi$

$$\cot(\operatorname{arc}\cot x) = x \quad \text{and} \quad \operatorname{arc}\cot(\cot y) = y.$$

If $|x| \ge 1$ and $-\pi/2 \le y < 0$ or $0 < y \le \pi/2$, then

$$\csc(\operatorname{arc}\csc x) = x \quad \text{and} \quad \operatorname{arc}\csc(\csc y) = y.$$

Example 1: Evaluate each function.

a. $\operatorname{arc}\cot(1) = \frac{\pi}{4}$

b. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

c. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$

$= \boxed{\frac{\pi}{6}}$

d. $\arctan\left(\sqrt{3}\right) = \boxed{\frac{\pi}{3}}$

e. $\arccos\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$

f. $\operatorname{arccsc}\left(-\sqrt{2}\right) = \boxed{-\frac{\pi}{4}}$

Example 2: Solve the equation for x.

$\tan(-\theta) = -\tan\theta$

$\tan\left(\arctan(2x-5)\right) = \tan(-1)$

$2x-5 = \tan(-1)$

$2x = 5\tan 1$

$\boxed{x = \left\{ -\frac{5\tan 1}{2} \right\}}$

Example 3: Write the expression in algebraic form. (HINT: Sketch a right triangle)

a. $\sec(\arctan 4x) = \sec\theta$

$= \frac{\sqrt{16x^2+1}}{1}$

$= \boxed{\sqrt{16x^2+1}}$

Let

$\tan(\theta) = \tan(\arctan 4x)$

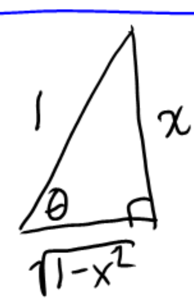
$\tan\theta = 4x = \frac{4x}{1}$

b. $\cos(\arcsin x)$

$= \cos\theta$

$= \frac{\sqrt{1-x^2}}{1}$

$= \boxed{\sqrt{1-x^2}}$



Let

$\sin(\theta) = \sin(\arcsin x)$

$\sin\theta = x = \frac{x}{1}$

Example 4: Differentiate with respect to *x*.

a. $y = \arcsin x$

b. $y = \arccos x$

c. $y = \arctan x$

d. $y = \operatorname{arc}\csc x$

e. $y = \operatorname{arc}\sec x$

f. $y = \operatorname{arc}\cot x$

See next page

What have we found out?!

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}[\arccos u] = -\frac{u'}{\sqrt{1-u^2}}$
3. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
4. $\frac{d}{dx}[\operatorname{arc}\cot u] = -\frac{u'}{1+u^2}$
5. $\frac{d}{dx}[\operatorname{arc}\sec u] = \frac{u'}{|u|\sqrt{u^2-1}}$
6. $\frac{d}{dx}[\operatorname{arc}\csc u] = -\frac{u'}{|u|\sqrt{u^2-1}}$

Ex: 4 $\quad u = f(x) \quad$ Differentiate w/respect to x

a) $\sin y = \sin(\arcsin u)$

$$\frac{d}{dx}\sin y = \frac{d}{dx}u$$

$$\frac{\cancel{(\cos y)}\frac{dy}{dx}}{\cancel{\cos y}} = \frac{du}{dx}\cdot\frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{du/dx}{\left(\frac{\sqrt{1-u^2}}{1}\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}}$$

d) $\csc y = \csc(\operatorname{arccsc} u)$

$$\frac{d}{dx}\csc y = \frac{d}{dx}u$$

$$\frac{(-\csc y\cot y)\frac{dy}{dx}}{-\csc y\cot y} = \frac{u'}{-\csc y\cot y}$$

$$\frac{dy}{dx} = -\frac{u'}{\left(\frac{u}{1}\right)\left(\frac{\sqrt{u^2-1}}{1}\right)}$$

$$\frac{dy}{dx} = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Example 5: Find the derivative of the function. Simplify your result to a single rational expression with positive exponents.

a. $\frac{d}{dt} f(t) = \frac{d}{dt}\left(\arcsin t^3\right)$

$$f'(t) = \frac{1}{\sqrt{1-(t^3)^2}} \cdot \frac{d}{dt}\left(t^3\right)$$

$$\boxed{f'(t) = \frac{3t^2}{\sqrt{1-t^6}}}$$

b. $\frac{d}{dx} g(x) = \frac{d}{dx}\left(\arcsin x + \arccos x\right)$

$$g'(x) = \frac{d}{dx}(\arcsin x) + \frac{d}{dx}(\arcsin x)$$

$$g'(x) = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\boxed{g'(x) = 0}$$

c. $\frac{d}{dx} y = \frac{d}{dx}\left(x \arctan 2x - \frac{1}{4}\ln\left(1+4x^2\right)\right)$

$$y' = \frac{d}{dx}(x\arctan 2x) - \frac{1}{4}\frac{d}{dx}\left(\ln(1+4x^2)\right)$$

$$y' = 1\arctan 2x + x \cdot \frac{1}{1+(2x)^2} \cdot \frac{d}{dx}(2x) - \frac{1}{4} \cdot \frac{1}{1+4x^2}\frac{d}{dx}\left(1+4x^2\right)$$

$$y' = \arctan 2x + \underbrace{\frac{2x}{1+4x^2}} - \frac{1}{\cancel{4}} \cdot \underbrace{\frac{1}{1+4x^2} \cdot \cancelto{2}{8}x} \rightarrow \boxed{y' = \arctan 2x}$$

d. $y = 25\arcsin\frac{x}{5} - x\sqrt{25-x^2}$

Example 6: Find an equation of the tangent line to the graph of the function $y=\frac{1}{2}\arccos x$ at the point $\left(-\frac{\sqrt{2}}{2},\frac{3\pi}{8}\right)$.

Find the slope (find $\frac{dy}{dx}$ and eval. at $x=-\frac{\sqrt{2}}{2}$)

$$\frac{d}{dx}y=\frac{d}{dx}\frac{1}{2}\arccos x$$

$$y'(x)=\frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{1}{2\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}}$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{1}{2\sqrt{1-\frac{2}{4}}}$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{1}{2\sqrt{\frac{4-2}{4}}}$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{1}{2\sqrt{\frac{2}{4}}}$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{1}{\frac{\cancel{2}\sqrt{2}}{\cancel{2}}}$$

$$y'\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{2}}{2}$$

Find the eq. of the tangent line to the graph at $\left(-\sqrt{2}/2, 3\pi/8\right)$

$$y-y_1=m(x-x_1)$$

$$\boxed{y-\frac{3\pi}{8}=-\frac{\sqrt{2}}{2}\left(x+\frac{\sqrt{2}}{2}\right)}$$