When you finish your homework you should be able to…

- π Use Sigma Notation to Write and evaluate a Sum
- π Understand the Concept of Area
- π Approximate the Area of a Plane Region
- π Find the Area of a Plane Region Using Limits

Warm-up: Evaluate the following limits.

1. $\lim_{x\to\infty}\frac{25x^2-5x+3}{x^2}$ $=\boxed{25}$

2. $\lim_{x\to\infty}\left(\frac{2x^3+6x^2}{3x^3}+10\right)$ $=\frac{2}{3}+10$ $=\boxed{\frac{32}{3}}$

SIGMA NOTATION

The sum of n terms $a_1,\ldots,a_n$ is written as $\sum_{i=1}^{n}a_i=a_1+a_2+a_3+\cdots+a_n$ where i is the **index of summation**, a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 1: Find the sum.

a. $\sum_{i=1}^{5}i$

$=1+2+3+4+5$

$a_1\ a_2\ a_3\ a_4\ a_5$

$=\boxed{15}$

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sum(seq(X,X,1,5,
1))
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exp. in the summation

with respect to

lower index

upper index

b. $\sum_{i=1}^{4}i^2$

$=(1)^2+(2)^2+(3)^2+(4)^2$

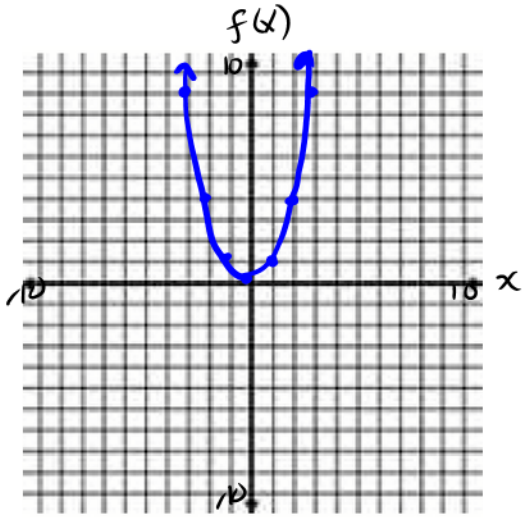
$=\boxed{30}$

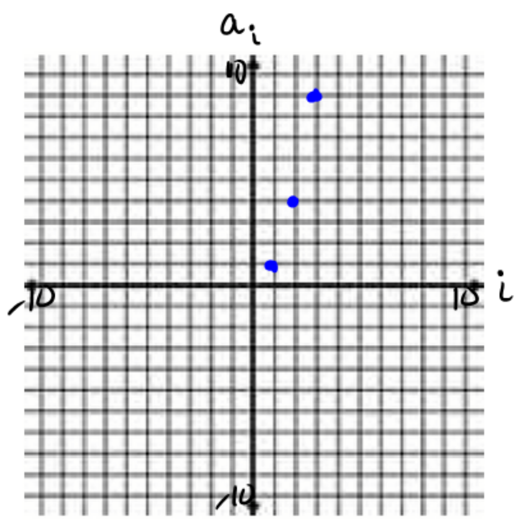
c. $\sum_{i=1}^{3}\sqrt{i}$

$=\boxed{\sqrt{1}+\sqrt{2}+\sqrt{3}}$

$f(x) = x^2$

$f(1) = (1)^2 = 1$





$a_i = i^2$

$a_1 = (1)^2 = 1$

$a_2 = (2)^2 = 4$

$a_i = i^2 = 1, 4, 9, 16, \ldots$

$\uparrow \quad \uparrow \quad \uparrow$

$a_1 \quad a_2 \quad a_3$

Summation Properties

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Example 2: Find the sum.

a. $\sum_{i=1}^{4} 2$

$= 2+2+2+2$

$= 2\cdot 4$

$= \boxed{8}$

b. $\sum_{i=1}^{3}(\underset{b_i}{\overset{a_i}{1 - i^2}})$

$= \sum_{i=1}^{3} 1 - \sum_{i=1}^{3} i^2$

$= (1+1+1) - (1+4+9)$

$= \boxed{-11}$

Theorem: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

you do not need to memorize

Example 3: Evaluate the following sums.

a. $\sum_{i=1}^{n}(3i - i^2)$

$= 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$

$= \boxed{3\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}}$

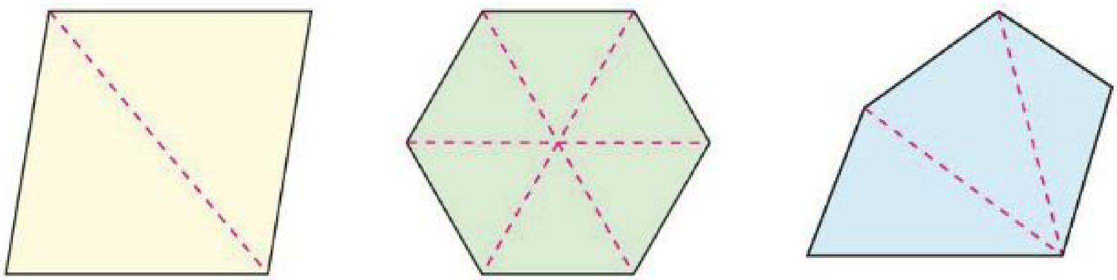
b. $\sum_{i=1}^{n}(6i + 4i^3)$

$= 6\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^3$

$= \boxed{6\frac{n(n+1)}{2} + 4\frac{n^2(n+1)^2}{4}}$

Area

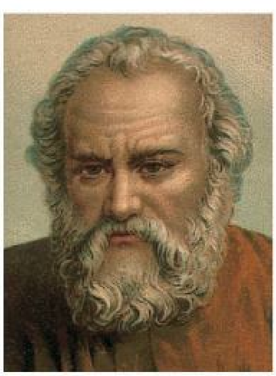
In Euclidean geometry, the simplest type of plane region is a __rectangle_____. The definition for the area of a rectangle is _$A=bh$_____.From this definition, you can develop formulas for the areas of many other plane regions such as triangles. To determine the area of a triangle, you can form a _rectangle______ whose area is __twice______ that of the _triangle_________. Once you know how to find the area of a triangle, you can determine the area of any _polygon______ by subdividing the polygon into __triangular_____ regions.



Parallelogram Hexagon Polygon

Figure 5.6

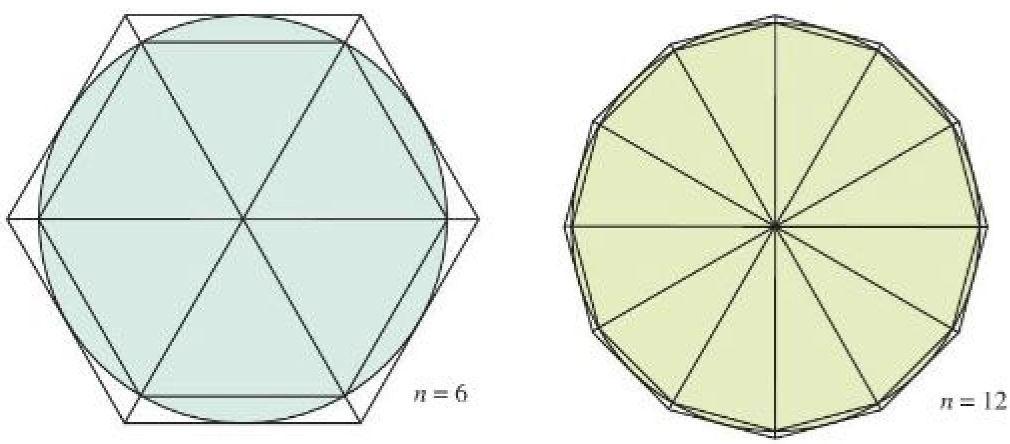
Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions by the _exhaustion____ method. Essentially, the method is a _limiting______ process in which the area is _squeezed___________ between two polygons—one _inscribed_______ in the region and one _circumscribed______ about the region.



ARCHIMEDES (287–212 B.C.)

Archimedes used the method of exhaustion to derive formulas for the areas of ellipses, parabolic segments, and sectors of a spiral. He is considered to have been the greatest applied mathematician of antiquity.

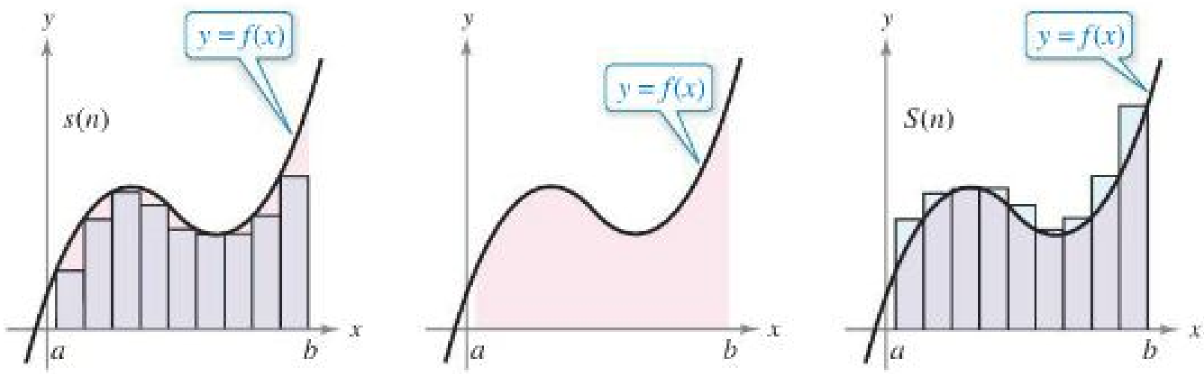
See LarsonCalculus.com to read more of this biography.



The exhaustion method for finding the area of a circular region

Figure 5.7

$$s(n) \le (\text{Area of region}) \le S(n)$$



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region.

Figure 5.11

Theorem: Limits of the Lower and Upper Sums

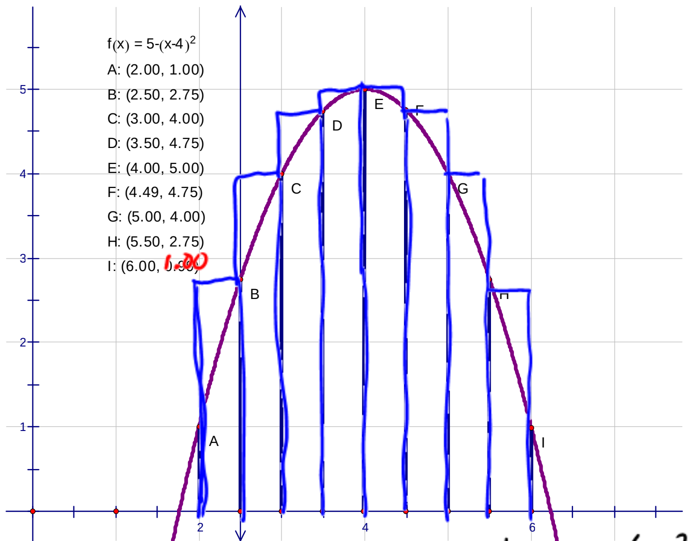
Let f be continuous and nonnegative on the interval $[a,b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = \dfrac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Example 4:

a. Find the upper sum from $x = 2$ to $x = 6$.



$a = 2,\ b = 6,\ n = 8$, so $\Delta x = \frac{b-a}{n} = \frac{6-2}{8} = \frac{1}{2}$

$$\text{Area} = \sum_{i=1}^{n} f(M_i)\Delta x$$

$$= \sum_{i=1}^{8} f(M_i)\cdot\frac{1}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{8} f(M_i)$$

$$= \frac{1}{2}\Big(f(2.5)+f(3)+f(3.5) + f(4)+f(4)+f(4.5) + f(5)+f(5.5)\Big)$$

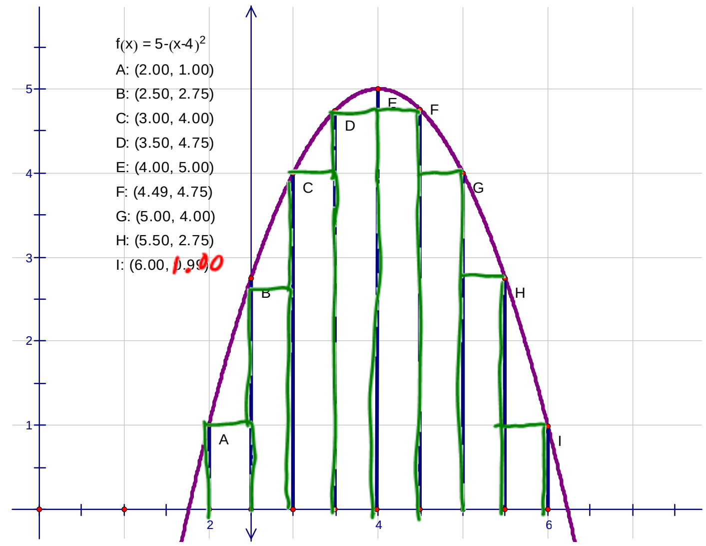
$$= \frac{1}{2}\,(2.75+4+4.75+5 + 5+4.75+4+2.75)$$

$$= \boxed{16.5 \text{ sq. units}}$$

$$\text{Area} \approx [S(n)+s(n)]\cdot\frac{1}{2}$$

$$= (16.5+12.5)\cdot\frac{1}{2} = 14.5 \text{ sq. units}$$

b. Find the lower sum from $x = 2$ to $x = 6$.



$$\text{Area} = \sum_{i=1}^{n} f(m_i)\Delta x$$

$$= \sum_{i=1}^{8} f(m_i)\frac{1}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{8} f(m_i)$$

$$= \frac{1}{2}\Big[f(2)+f(2.5)+f(3)+f(3.5) \cancel{f(3.5)} + f(4.5)+f(5)+f(5.5) + f(6)\Big]$$

$$= \frac{1}{2}\,(1+2.75+4+4.75+4.75 + 4+2.75+1)$$

$$= \boxed{12.5 \text{ sq. units}}$$

Definition of an Area in the Plane

Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x=a$ and $x=b$ is

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x, \qquad x_{i-1}\le c_i \le x_i \qquad \text{where } \Delta x = \frac{b-a}{n},$$

right endpoint: $c_i = a + i\Delta x$, left endpoint: $c_i = a+(i-1)\Delta x$

Example 5: Find the area of the region bounded by the graph $f(x)=x^3$, the x-axis, and the vertical lines $x=0$ and $x=1$.

$a=0,\ b=1$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

right endpoint to find c_i:

$a+i\Delta x = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$

$f(c_i) = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^3 = \frac{i^3}{n^3}$

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\frac{1}{n^{\cancel{4}\,2}}\cdot\frac{\cancel{n^2}(n+1)^2}{4}$$

$$= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2}$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right)$$

$$= \frac{1}{4}+0+0$$

$$= \boxed{\frac{1}{4}\text{ sq. unit}}$$