When you are done with your homework you should be able to…

- π Find the derivative of a function using the constant rule
- π Find the derivative of a function using the power rule
- π Find the derivative of a function using the constant multiple rule
- π Find the derivative of a function using the sum and difference rules
- π Find the derivative of the sine function and of the cosine function
- π Use derivatives to find rates of change

$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Warm-up: Find the following derivatives using the limit definition of the derivative.

1. $f(x)=2$

$f'(x)=\lim_{\Delta x\to 0}\frac{2-2}{\Delta x} \rightarrow f'(x)=\lim_{\Delta x\to 0}\frac{0}{\Delta x} \rightarrow f'(x)=\lim_{\Delta x\to 0} 0$

$\boxed{f'(x)=0}$

2. $f(x)=x^2$

$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$

$f'(x)=\lim_{\Delta x\to 0}\frac{[(x+\Delta x)+x][(x+\Delta x)-x]}{\Delta x}$

$f'(x)=\lim_{\Delta x\to 0}\frac{(2x+\Delta x)\cancel{\Delta x}}{\cancel{\Delta x}}$

D.S.

$f'(x)=2x+0$

$\boxed{f'(x)=2x}$

3. $f(x)=\cos x$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\lim_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x}=0$

$f'(x)=\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}$

$f'(x)=\lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x-\sin x\sin\Delta x-\cos x}{\Delta x}$

$f'(x)=\lim_{\Delta x\to 0}\frac{-\cos x(1-\cos\Delta x)-\sin x\sin\Delta x}{\Delta x}$

$f'(x)=\lim_{\Delta x\to 0}\frac{-\cos x(1-\cos\Delta x)}{\Delta x}-\lim_{\Delta x\to 0}\frac{\sin x\sin\Delta x}{\Delta x}$

$$f'(x) = \left(\lim_{\Delta x \to 0} -\cos x\right)\underbrace{\left(\lim_{\Delta x \to 0} \frac{1-\cos \Delta x}{\Delta x}\right)}_{\text{special trig lim}} - \left(\lim_{\Delta x \to 0} \sin x\right)\underbrace{\left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}\right)}_{\text{special trig lim.}}$$

$$f'(x) = (-\cos x)(0) - (\sin x)(1)$$

$$\boxed{f'(x) = -\sin x}$$

THEOREM: THE CONSTANT RULE

The derivative of a constant function is zero. That is, if c is a real number, then

$$\frac{d}{dx}[c]=0$$

Hmmm…isn't this theorem the equivalent of saying that the slope of a horizontal line is zero?

Example 1: Find the derivative of the function $g(x)=6$.

$\boxed{g'(x)=0}$

THEOREM: THE POWER RULE

If n is a rational number, then the function $f(x)=x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

For f to be differentiable at $x=0$, n must be a number such that x^{n-1} is defined on an interval containing zero.

Example 2: Find the following derivatives.

a. $f(x)=x^5$

$f'(x)=5x^{5-1}$

$\boxed{f'(x)=5x^4}$

b. $f(x)=x^{1/2}$

$f'(x)=\frac{1}{2}x^{1/2-1}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$\boxed{f'(x)=\frac{1}{2x^{1/2}}}$

c. $f(x)=x^{-5/3}$

$f'(x)=-\frac{5}{3}x^{-5/3-1}$

$f'(x)=-\frac{5}{3}x^{-8/3}$

$\boxed{f'(x)=-\frac{5}{3x^{8/3}}}=-\frac{5}{3x^2\sqrt[3]{x^2}}$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$$

$$\frac{d}{dx}\left[cf(x)\right] = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} c \cdot \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= cf'(x) //$$

Example 3: Find the slope of the graph of $f(x) = 2x^3$ at

$$f'(x) = 2 \cdot \frac{d}{dx} x^3 \rightarrow f'(x) = 2 \cdot 3x^2$$

$$f'(x) = 6x^2$$

a. $x = 2$

$$f'(2) = 6(2)^2 = 24$$

b. $x = -6$

$$f'(-6) = 6 \cdot (-6)^2 = 216$$

c. $x = 0$

$$f'(0) = 6(0)^2 = 0$$

THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right] = f'(x)-g'(x)$$

Example 4: Find the equation of the line tangent to the graph of $f(x) = x - \sqrt{x}$ at $x = 4$.

① slope (derivative)

$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left(x - x^{1/2}\right)$

$f'(x) = \left(\frac{\partial}{\partial x} x\right) - \left(\frac{\partial}{\partial x} x^{1/2}\right)$

$f'(x) = 1 - \frac{1}{2} x^{-1/2}$

$f'(4) = 1 - \frac{1}{2}(4)^{-1/2}$

$f'(4) = 1 - \frac{1}{2} \cdot \frac{1}{2}$

$f'(4) = \frac{3}{4} = m_{tan}$

② $f(4) = 4 - \sqrt{4} = 2$

③ $y - y_1 = m(x - x_1)$

$\boxed{y - 2 = \frac{3}{4}(x - 4)}$

THEOREM: DERIVATIVES OF THE SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Example 5: Find the derivative of the following functions:

a. $f(x) = \frac{\sin x}{6}$

$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left(\frac{1}{6} \sin x\right)$

$f'(x) = \frac{1}{6}\left(\frac{\partial}{\partial x} \sin x\right)$

$\boxed{f'(x) = \frac{1}{6} \cos x}$

b. $\frac{\partial}{\partial \theta} r(\theta) = \frac{\partial}{\partial \theta}(5\theta - 3\cos\theta)$

$r'(\theta) = \frac{\partial}{\partial \theta} 5\theta - \frac{\partial}{\partial \theta} 3\cos\theta$

$r'(\theta) = 5 \frac{\partial}{\partial \theta} \theta - 3 \frac{\partial}{\partial \theta} \cos\theta$

$r'(\theta) = 5 \cdot 1 - 3(-\sin\theta)$

$\boxed{r'(\theta) = 5 + 3\sin\theta}$

RATES OF CHANGE

We have seen how the derivative is used to determine __slope__. The derivative may also be used to determine the __rate__ of __change__ of one __variable__ with respect to another.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the __right__ or __upwards__ is considered to be in the positive direction, and movement to the left or downwards is considered to be in the __negative__ direction.

THE POSITION FUNCTION is denoted by s and gives the position (relative to the origin) of an object as a function of time. If, over a period of time Δt, the object changes its position by $\Delta s = s(t+\Delta t) - s(t)$, then, by the familiar formula

$$\text{rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

Example 6: A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. The position function for free-falling objects measured in feet is $s(t) = -16t^2 + v_0 t + s_0$. $s(t) = -16t^2 - 22t + 220$

What is its velocity after 3 seconds?

$s'(t) = v(t) = -16 \cdot 2t - 22 \cdot 1 + 0$

$v(t) = -32t - 22$

$v(3) = -32(3) - 22$

$\boxed{v(3) = -118 \text{ ft/s}}$

$220 - 108 = 112$

What is its velocity after falling 108 feet?

$112 = -16t^2 - 22t + 220$

$\frac{0}{-2} = \frac{-16t^2 - 22t + 108}{-2}$

$0 = 8t^2 + 11t - 54$

$t = \frac{-11 \pm \sqrt{(11)^2 - 4(8)(-54)}}{2(8)}$

$t = \frac{-11 \pm \sqrt{121 + 1728}}{16}$

$t = \frac{-11 \pm \sqrt{1849}}{16}$

$t = \frac{-11 \cancel{\pm} 43}{16}$

$t = 2 \text{ sec}$

$v(t) = -32t - 22$

$v(2) = -32(2) - 22$

$\boxed{v(2) = -86 \text{ ft/s}}$