When you are done with your homework you should be able to...
- π Determine continuity at a point and continuity on an open interval
- π Determine one-sided limits and continuity on a closed interval
- π Use properties of continuity
- π Understand and use the Intermediate Value Theorem

DEFINITION OF CONTINUITY

CONTINUITY AT A POINT: A function f is **continuous at c** if the following **three** conditions are met.

1. $f(c)$ is defined.

2. $\lim\limits_{x \to c} f(x)$ exists.

3. $\lim\limits_{x \to c} f(x) = f(c)$.

Use the 3 condition for cont. at a point to determine if f is continuous at $x = 3$.

1) $f(3) = 1$ ✓

2) $\lim\limits_{x \to 3} f(x) = 3$ ✓

3) $\lim\limits_{x \to 3} f(x) \neq f(3)$

Fails

f is not cont. at $x = 3$
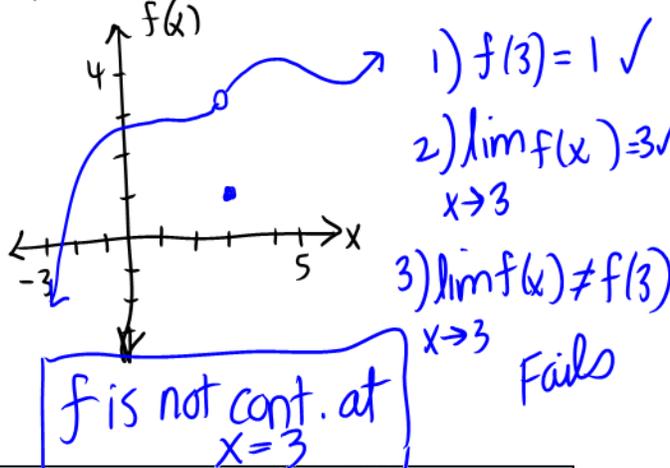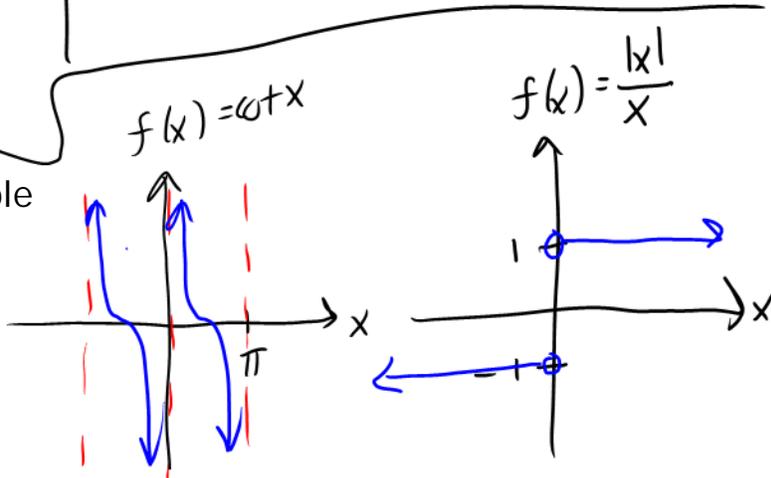
CONTINUITY ON AN OPEN INTERVAL:

A function is **continuous on an open interval (a,b)** if it is continuous at **each point in the interval**. A function that is continuous on the entire real line is **everywhere continuous**.
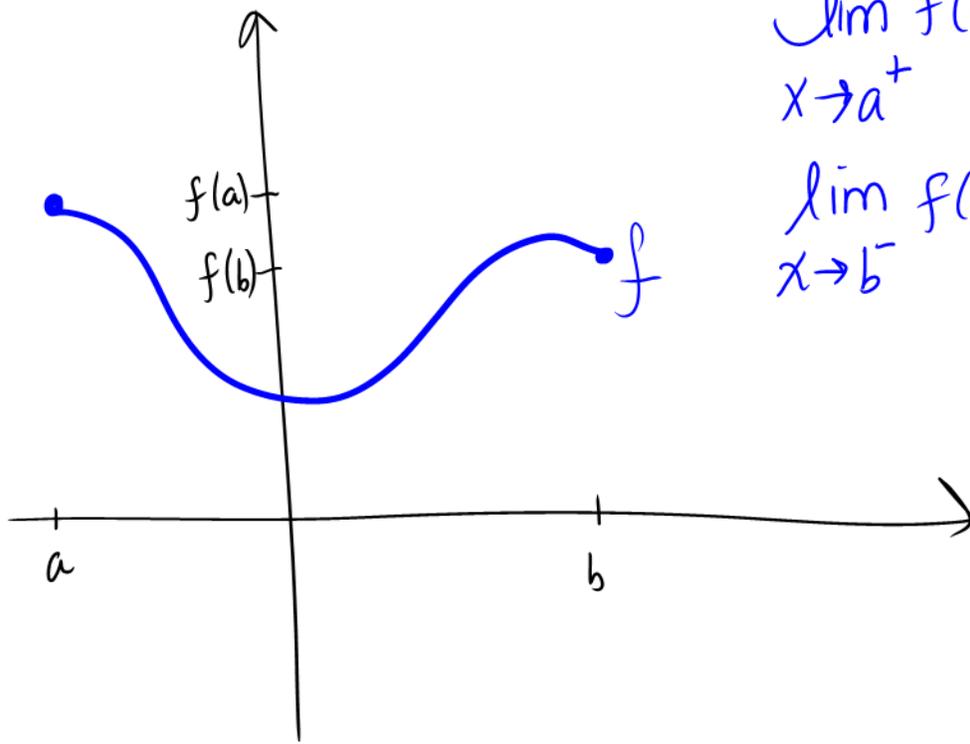
1. Draw the graph of the following functions with the given characteristics on the open interval from a to b:

   a. The function has a removable discontinuity at $x = 0$

   $f(x) = \cot x$
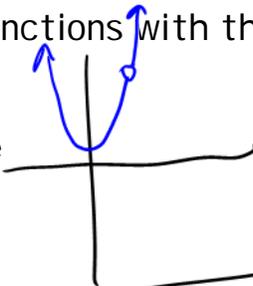
   b. The function has a nonremovable discontinuity at $x = 0$

   $f(x) = \dfrac{|x|}{x}$

$f(a)$

$f(b)$

a

b

f

$$\lim_{x \to a^+} f(x) = f(a)$$

$$\lim_{x \to b^-} f(x) = f(b)$$

THEOREM: THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

CONTINUITY ON A CLOSED INTERVAL:

A function f is **continuous on the closed interval [a,b]** if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right at a** and **continuous from the left at b**.

2. Graph each function and use the definition of continuity to discuss the continuity of each function.

f is cont. on:
$(-\infty, -2) \cup (-2, \infty)$
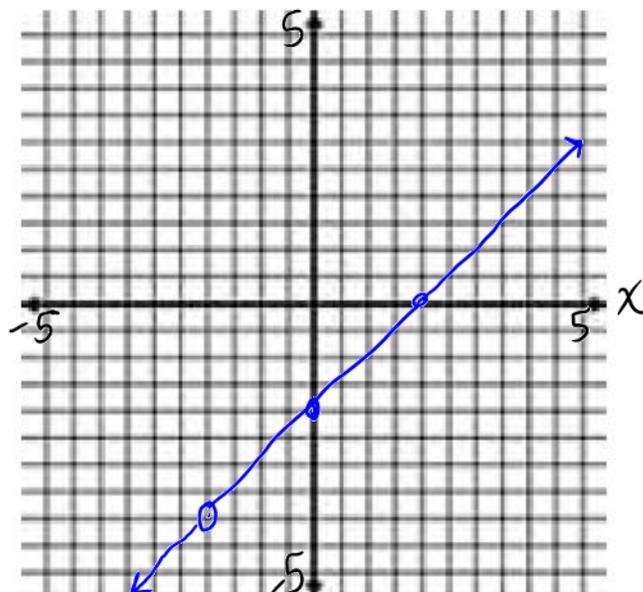$f(x)$

a. $f(x) = \dfrac{x^2 - 4}{x + 2}$

$f(x) = \dfrac{(x+2)(x-2)}{x+2}$

$f(x) = x - 2, \quad x \neq -2$

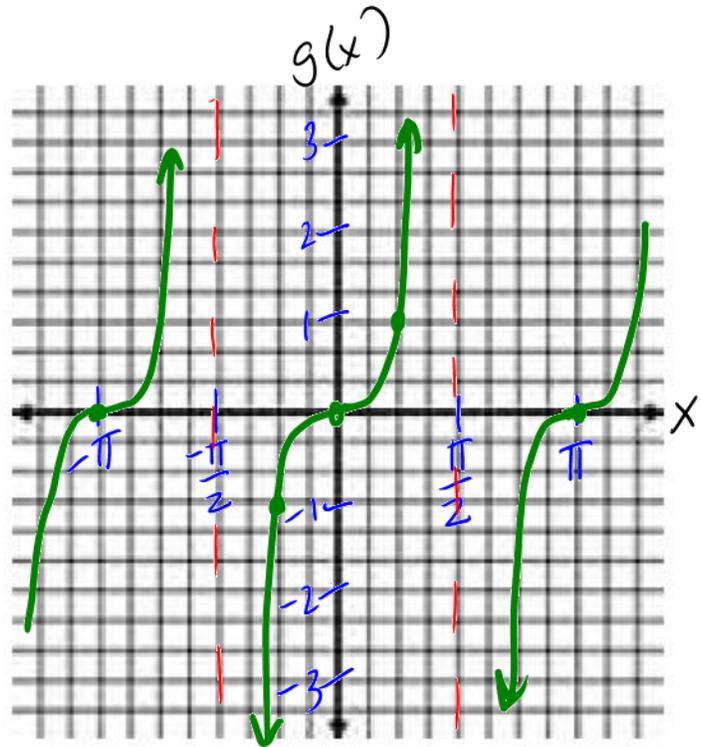$f(-2)$ DNE, but there's a hole

at $f(-2) = -2-2 = -4$

$f(0) = -2$

b. $g(x) = \tan x$

g is cont. on:

$\{ x \mid x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \}$
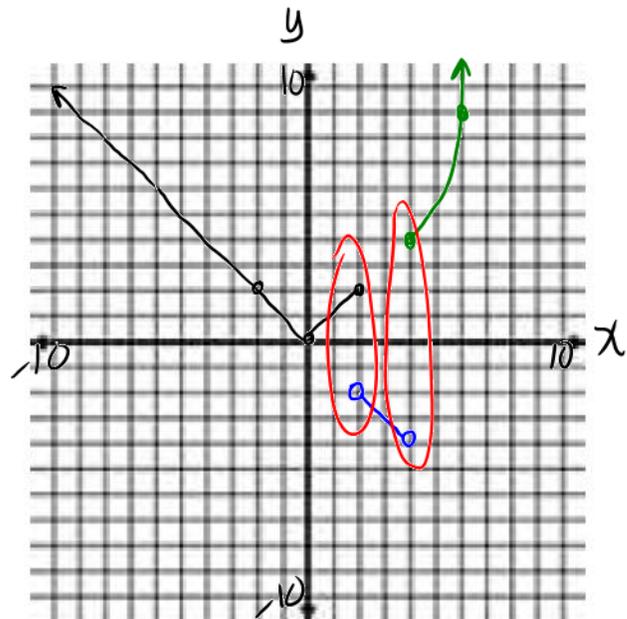

$g(x)$

c.

$$y = \begin{cases} |x|, & x \leq 2 \\ -x, & 2 < x < 4 \\ \dfrac{x^2}{4}, & x \geq 4 \end{cases}$$

On what intervals is y cont.?

$(-\infty, 2) \cup (2, 4) \cup (4, \infty)$

$y(2) = |2| = 2$

open circle at $(2, -2)$ middle interval
open circle at $(4, -4)$

$y(4) = \dfrac{4^2}{4} = 4$

$y(6) = \dfrac{6^2}{4} = 9$

THEOREM: PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$ then the following functions are also continuous at c.

1. Scalar multiple: bf

2. Sum or difference: $f \pm g$

3. Product: fg

4. Quotient: $\dfrac{f}{g}, \ g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + x_0$

2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}, \ q(x) \neq 0$

3. Radical functions: $f(x) = \sqrt[n]{x}$

4. Trigonometric functions: $\sin x, \ \cos x, \ \tan x, \ \cot x, \ \sec x, \ \csc x$

3. Explain why the following functions are continuous at every point in their domains.

   a. $f(x) = \sqrt{x} - \tan x$

   Properties used:

b. $f(x) = \dfrac{5-x}{x \sin x}$

Properties used:

THEOREM: CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$

then the composite function $(f \circ g)(x) = f(g(x))$

is continuous at c.

4. Explain why the following functions are continuous at every point in their domains.

a. $f(x) = \sqrt[3]{x^2 - 8x + 1}$
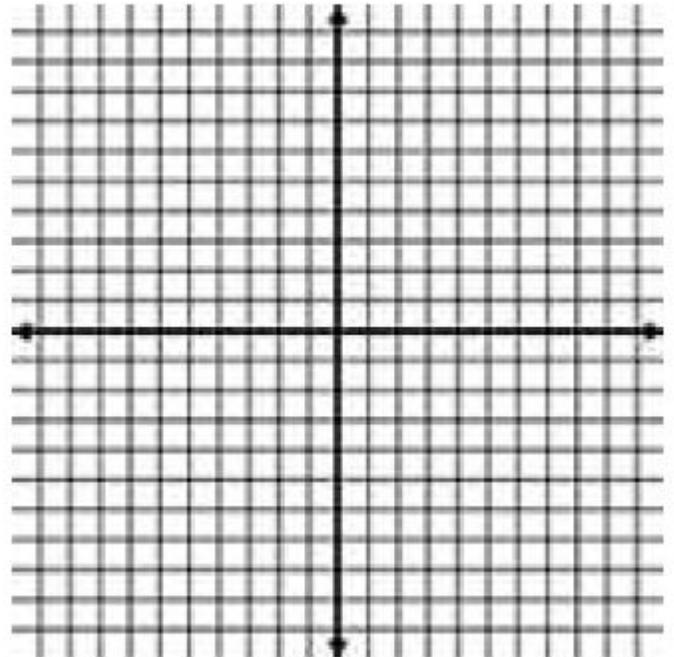
Properties used:

b. $g(x) = \tan(2x^2)$

Properties used:

THEOREM: THE INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is **at least** one number c in $[a,b]$ such that $f(c)=k$.

5. Consider the function $f(x)=\cos 2x$ on the closed interval $[0,\pi]$.

   a. Sketch the graph of f by hand.

   b. State the reason why we can apply the intermediate value theorem (IVT).

   c. Use the IVT to find c such that $f(c)=\dfrac{1}{2}$.