7.4: Arc length and area of a surface of revolution

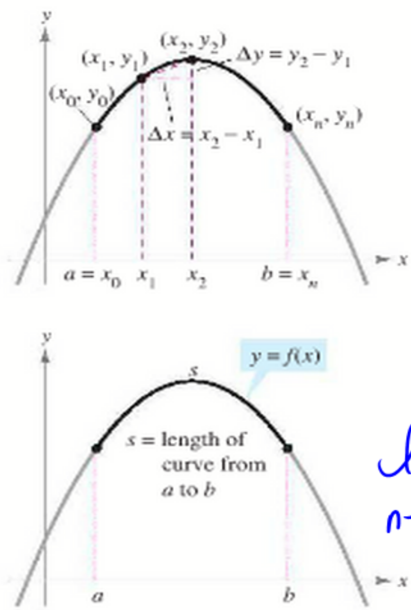

Figure 7.37

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$a=x_0<x_1<x_2\cdots<x_{n-1}<x_n=b$$

Consider x_i-x_{i-1} and y_i-y_{i-1} ...

$x_i-x_{i-1}=\Delta x_i$ and $y_i-y_{i-1}=\Delta y_i$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2+\left[(\Delta y_i)^2\right]\left[\frac{(\Delta x_i)^2}{(\Delta x_i)^2}\right]}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(\Delta x_i)^2\left[1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right]}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

$$=\int_a^b\sqrt{1+(y')^2}\,dx$$

MATH 150/GRACEY NAME________________________________

WORKSHEET/CHAPTER 7.4-7.5

ARC LENGTH AND AREA OF A SURFACE OF REVOLUTION

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx, \ y \text{ is a function of } x,$$

If $x = g(y)$ on the interval $[c,d]$, then the arc length of g between c and d is

$$s = \int_c^d \sqrt{1+\left[g'(y)\right]^2}\,dy, \ x \text{ is a function of } y$$

1. Find the arc length of the graph of the function $y = \frac{3}{2}x^{2/3}$ over the interval $[1,4]$.

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = x^{-1/3}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(x^{-1/3}\right)^2}$$

$$= \sqrt{1+x^{-2/3}}$$

$$= \sqrt{1+\frac{1}{x^{2/3}}}$$

$$= \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}$$

$$= \frac{\sqrt{x^{2/3}+1}}{x^{1/3}}$$

$$S = \int_1^4 \frac{\sqrt{x^{2/3}+1}}{x^{1/3}}\,dx$$

$$u = x^{2/3}+1$$

$$\frac{du}{dx} = \frac{2}{3}x^{-1/3} \rightarrow dx = \frac{du}{\frac{2}{3}x^{-1/3}}$$

$$dx = \frac{3}{2}x^{1/3}\,du$$

upper: $4^{2/3}+1$

lower: $1^{2/3}+1 = 2$

$$S = \int_2^{4^{2/3}+1} \frac{u^{1/2}}{\cancel{x^{1/3}}} \cdot \frac{3}{2}\cancel{x^{1/3}}\,du$$

$$S = \cancel{\frac{3}{2}} \cdot \cancel{\frac{2}{3}}\, u^{3/2} \Big|_2^{4^{2/3}+1}$$

$$S = \left[\left(4^{2/3}+1\right)^{3/2} - 2^{3/2}\right] \text{units}$$

2. Find the arc length of the graph of the function $y = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $\left[\frac{1}{2}, 2\right]$.

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^4}{4} - \cancel{2}\left(\frac{x^{\cancel{2}}}{\cancel{2}}\right)\left(\frac{1}{2\cancel{x^2}}\right) + \frac{1}{4x^4}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} + 1$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}$$

$$\left(1 + \frac{dy}{dx}\right)^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$S = \int_{1/2}^{2} \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$

$$S = \int_{1/2}^{2} \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$S = \frac{1}{2}\int_{1/2}^{2} \left(x^2 + x^{-2}\right) dx$$

$$S = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2}$$

$$S = \frac{1}{2}\left[\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1/8}{3} - \frac{1}{1/2}\right)\right]$$

$$S = \frac{1}{2}\left[\left(\frac{16-3}{6}\right) - \left(\frac{1}{24} - 2\right)\right]$$

$$S = \frac{1}{2}\left(\frac{13 \cdot 4}{6 \cdot 4} - \left(-\frac{47}{24}\right)\right)$$

$$S = \frac{1}{2}\left(\frac{99}{24}\right)$$

$$\boxed{S = \frac{99}{48} \text{ units}}$$

Let $y = f(x)$ have a continuous derivative on the interval $[a,b]$. The area S of the surface of revolution formed by revolving the graph of about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx,$$

y is a function of x, where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c,d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + \left[g'(y)\right]^2}\, dy,$$

x is a function of y, where $r(y)$ is the distance between the graph of g and the axis of revolution.

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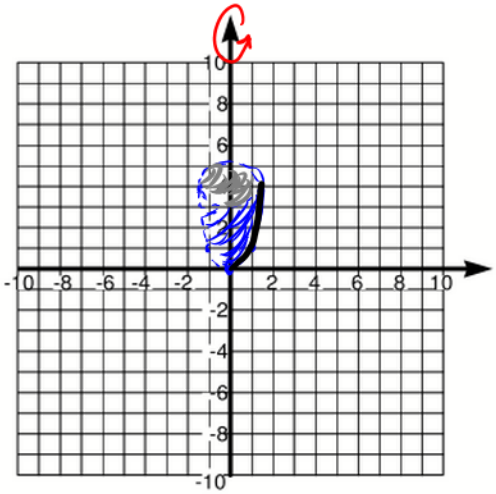
3. Find the **area of the surface** formed by revolving the graph of $f(x) = x^2$ on the interval $\left[0, \sqrt{2}\right]$ about the y-axis.

$y=x^2, [0,\sqrt{2}]$



$r(x)=x \quad , y=x^2 \rightarrow \frac{dy}{dx}=2x$

$$S=2\pi\int_a^b r(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S=2\pi\int_0^{\sqrt{2}} x\sqrt{1+(2x)^2}\,dx$$

$$S=2\pi\int_0^{\sqrt{2}} x\sqrt{1+4x^2}\,dx$$

$u=1+4x^2$

$\frac{du}{dx}=8x$

$dx=\frac{du}{8x}$

$$S=\cancel{2}\pi\int_1^9 \cancel{x}\cdot u^{1/2}\,\frac{du}{\cancel{8}_{4}\cancel{x}}$$

upper: $1+4(\sqrt{2})^2=9$

lower: $1+4(0)^2=1$

$$S=\frac{\pi}{4}\int_1^9 u^{1/2}\,du$$

$$S=\frac{\pi}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^9$$

$$S=\frac{\pi}{6}\left(9^{3/2}-1^{3/2}\right)$$

$$S=\frac{\pi}{6}(27-1)$$

$$S=\frac{\pi}{6}(26)$$

$$\boxed{S=\frac{13\pi}{3}\text{ sq. units}}$$